MMAT 5120 Topics in Geometry

Lecture 8

Euclid's Postulates (cont'd)

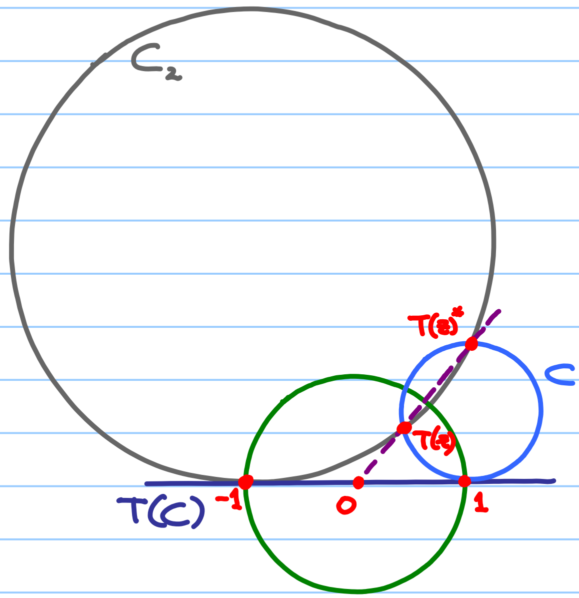
Thm Euclid's Postulate 5 is false in hyperbolic geometry.
More precisely, for any point z not on a hyperbolic straight line C_0, there are 2 hyperbolic straight lines parallel to C_0 and passing through z.

Pf: For any hyperbolic straight line C_0, $\exists$ transformation $T \in H$ such that $T(C_0) = x$-axis. If $z \in \mathbb{D}$ and $z \notin C_0$, then $T(z) \in \mathbb{D}$ is a pt not lying on the x-axis.
A hyperbolic straight line is parallel to the x-axis if they

share an ideal point, meaning that it has to pass through either 1 or -1 (intersection pts of x-axis and $C = \partial\mathbb{D}$).
But Lemma 2 in Lecture 7 says that any hyperbolic straight line passing through $T(z)$ must also passes through $T(z)^* = T(z^*)$.
So there are exactly 2 choices:

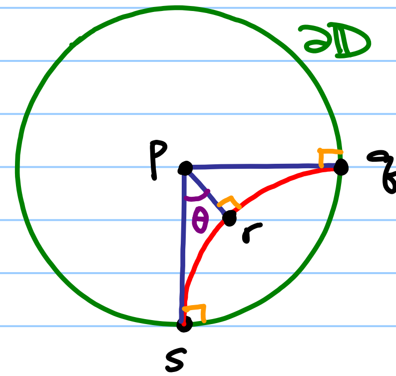
C_1 passing thru $1, T(z), T(z)^*$ and

C_2 passing thru $-1, T(z), T(z)^*$



Hence, there are precisely 2 hyperbolic straight lines, namely, $T^{-1}(C_1)$ and $T^{-1}(C_2)$, parallel to C_0 and passing through z. #

Angle of parallelism



Consider a hyperbolic straight line $\overline{srq}$ and a point $p \in \mathbb{D}$ not on it.

By applying a transformation if necessary, we may assume that p is at the origin.

Note that the 2 lines parallel to $\overline{srq}$ thru p are $\overline{ps}$ and $\overline{pq}$. The angle θ between one of these lines (say $\overline{ps}$) and the perpendicular $\overline{pr}$ is called the angle of parallelism.

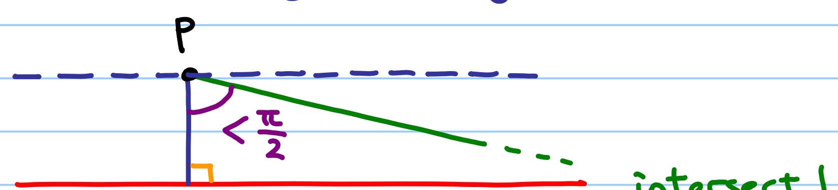
Rmk The angle of parallelism is always acute.

Consider a ray passing through p (i.e. a hyperbolic straight line).

If the angle it makes with $\overline{pr}$ is

$$\left\{\begin{array}{l} < \theta \\ = \theta \\ > \theta \end{array}\right. \quad \text{then it} \quad \left\{\begin{array}{l} \text{intersects } \overline{srq} \\ \text{is parallel to } \overline{srq} \\ \text{is hyperparallel to } \overline{srq} \end{array}\right.$$

Compare with Euclidean geometry :

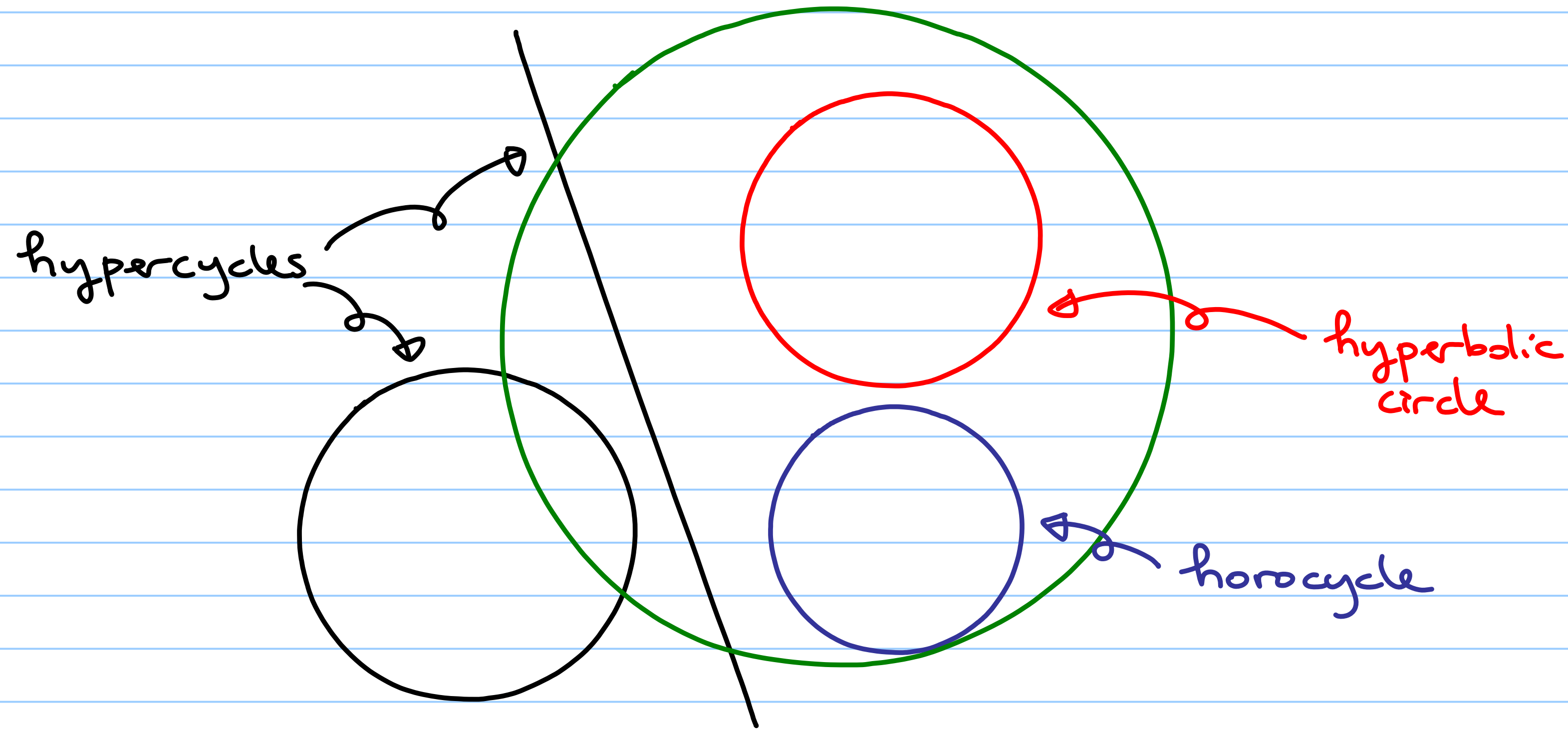


§ Cycles

Def Let C be a Euclidean circle or Euclidean straight line which has nonempty intersection with $\mathbb{D}$. Suppose that C is not orthogonal to $\partial\mathbb{D}$. Then C is called a cycle.

There are 3 types:

1. If C is entirely contained in $\mathbb{D}$, then C is called a hyperbolic circle.
2. If C is tangent to $\partial\mathbb{D}$, then C is called a horocycle.
3. If C intersects $\partial\mathbb{D}$ at two distinct points (at angles $\neq \frac{\pi}{2}$), then C is called a hypercycle.


hypercycles
hyperbolic circle
horocycle

Lemma Let $T \in M$ be a Möbius transformation, and C be a circle such that $T(C) = C$. If z is a fixed pt of T, then the pt z^* symmetric w.r.t. C is also fixed by T

Pf: This is because we have $T(z^*) = T(z)^* = z^*$. #

In particular, for $T \in H$, since $T(\partial \mathbb{D}) = \partial \mathbb{D}$, there are 3 possible cases:

① T has 1 fixed pt $p \in \mathbb{D}$ and 1 fixed point $p^* \in \mathbb{C} \setminus \mathbb{D}$.

② T has 2 fixed pts on $\partial \mathbb{D}$.

③ T has 1 fixed pt on $\partial \mathbb{D}$.

We will see that these 3 cases exactly correspond to the 3 types of cycles in $\mathbb{D}$.