MMAT 5120 Topics in Geometry

Lecture 4

§ Möbius geometry

Def Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2$ be the extended complex plane, and let M be the set of transformations of the form

$$w = T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and the determinant of T, $ad-bc \neq 0$.

Such a transformation is called a Möbius transformation (or linear fractional transformation).

The pair $(\hat{\mathbb{C}}, M)$ models Möbius geometry.

<u>Rmk</u> Möbius transformations include all rotations, translations, homothetic transformations and the inversion. (How?)

Conversely,

$$w = T(z) = \frac{az+b}{cz+d} = \begin{cases} \frac{a}{c} - \frac{ad-bc}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right) & \text{if } c \neq 0 \\ \frac{a}{d} z + \frac{b}{d} & \text{if } c = 0 \end{cases}$$

In any case, T is a composition of rotations, translations, homothetic transformations and the inversion.

(We say M is generated by rotations, translations, homothetic transformations and the inversion.)

Pf of the fact that $(\hat{\mathbb{C}}, M)$ is a geometry:

(o) $\forall$ Möbius transformation $T(z) = \dfrac{az+b}{cz+d}$, we can set

$$T(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases} \quad \text{and} \quad T\left(-\frac{d}{c}\right) = \infty \quad \text{if } c \neq 0.$$

Then T defines a transformation on $\hat{\mathbb{C}}$.

(i) By setting $a=d=1$ and $b=c=0$ (so $ad-bc=1\neq 0$), we see that $Id_{\hat{\mathbb{C}}} \in M$.

(iii) Let $T(z) = \dfrac{az+b}{cz+d}$, $S(z) = \dfrac{ez+f}{gz+h}$

(with $ad-bc \neq 0$ $eh-gf \neq 0$)

be two Möbius transformations.

We can represent them by 2×2 complex matrices:

$$T \longleftrightarrow A_T := \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad S \longleftrightarrow A_S := \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Then the composition $T\circ S$ is given by the Möbius transformation represented by the matrix product $A_T \cdot A_S$. (check!)

(iv) Given $T \in M$ represented by a matrix $A_T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is given by $T^{-1} \in M$ represented by $A_T^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (which exists because $\det A_T = ad-bc \neq 0$, and this is why we need this condition).

We conclude that M is a transformation group. #

Rmk Note that for any $k \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the matrix $k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$ represents $T(z) = \dfrac{kaz + kb}{kcz + kd} = \dfrac{az+b}{cz+d} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

So all the matrices kA_T, $k \in \mathbb{C}^*$ correspond to the same transformation! To remedy this, we may normalize the matrix representation by requiring that $\boxed{\det A_T = ad - bc = 1}$.

This reduces the ambiguity to $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow T(z) = \frac{az+b}{cz+d}$.

Hence the group M of Möbius transformations is represented by the matrix group $SL(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{C}, ad-bc=1 \right\}$, called the special linear group, up to $\pm I$ (where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$).

More precisely, we have

$$M \cong SL(2,\mathbb{C})/\{\pm I\} =: \underset{\substack{\uparrow \\ \text{projective special linear group}}}{PSL(2,\mathbb{C})}$$

(quotient group)

Fixed pts of Möbius transformations

A fixed point of a transformation T is a point $z \in \hat{\mathbb{C}}$ s.t. $T(z) = z$.

e.g. A translation $z \mapsto z+b$ $(b \neq 0)$ has only one fixed pt $\infty \in \hat{\mathbb{C}}$.

A rotation $z \mapsto e^{i\theta} z$ $(0 < \theta < 2\pi)$ has two fixed pts 0 & $\infty \in \hat{\mathbb{C}}$.

For a Möbius transformation $T(z) = \frac{az+b}{cz+d}$ (with $ad-bc \neq 0$),

$$T(z) = z \iff \frac{az+b}{cz+d} = z \iff cz^2 + (d-a)z - b = 0 \quad - (*)$$

There are 3 cases :

① $c \neq 0$: $(*)$ has 1 or 2 sols, and ∞ is not fixed as $T(\infty) = \frac{a}{c}$,
so T has 1 or 2 fixed pts.

② $c = 0$, $a \neq d$: $(*)$ has 1 sol $z = \frac{b}{d-a}$ and $T(\infty) = \infty$,
so T has 2 fixed pts: $\frac{b}{d-a}, \infty \in \hat{\mathbb{C}}$.

③ $c = 0$, $a = d \neq 0$: Then $T(z) = z + \frac{b}{d}$.
so either T has a unique fixed pt $\infty \in \hat{\mathbb{C}}$ (when $b \neq 0$)
or $T = Id_{\hat{\mathbb{C}}}$ (when $b = 0$) which fixes all pts in $\hat{\mathbb{C}}$.

From this we conclude that

Lemma If a Möbius transformation $T \neq \mathrm{Id}_{\hat{\mathbb{C}}}$, then it has either one or two fixed pts. In particular, if T has 3 or more fixed pts, then we must have $T = \mathrm{Id}_{\hat{\mathbb{C}}}$.

Next time, we'll prove:

Thm (The Fundamental Theorem of Möbius Geometry)
For any distinct triple $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and another distinct triple $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, $\exists!$ Möbius transformation $T \in M$ s.t. $T(z_i) = w_i$ for $i = 1, 2, 3$.