THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Homework 3 solution
7th April 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. As we have seen in Q2 of lectures 9 and 10 practice problems set, a hyperbolic circle is really a circle with respect to hyperbolic distance, i.e. $C = \{z \in \mathbb{D} | \, d(z, z_0) = r\}$ where $z_0 \in \mathbb{D}$ and $r > 0$. Now pick any point $w \in C$, then the hyperbolic straight line L passing through both z_0 and w would be perpendicular to C itself, this can be seen by simply passing the picture to the origin, i.e. we consider a transformation S taking z_0 to 0, then $S(C)$ would become a circle centered at the origin, and the hyperbolic straight lines are straight lines through the origin and it is easy to see the angle they make is $\pi/2$. Now L being a hyperbolic straight line also means that if we extends it towards the ideal points, L would also be perpendicular to $\partial\mathbb{D}$ by definition. So L satisfies the description that it is perpendicular to C and $\partial\mathbb{D}$, and it can be extended to form a Euclidean circle. (Except in the case when $z_0 = 0$, where we will get L being straight lines, then they are still clines regardless.)

   Conversely, if L is a line perpendicular to both C and $\partial\mathbb{D}$, then by definition the portion of L within $\mathbb{D}$ would be a hyperbolic straight line, and again passing to the case where $z_0 \mapsto 0$, we see that all such lines actually intersect the same point 0.

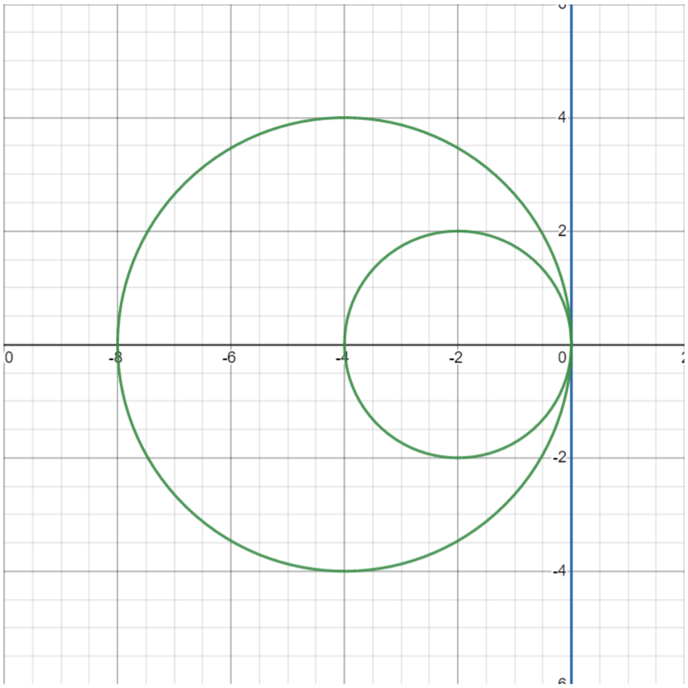
   Therefore we have that the family of lines perpendicular to C and $\partial\mathbb{D}$ is actually just the family of hyperbolic straight line passing through $z_0 = p$. Now by lecture 7 lemma 2, these are precisely all lines would pass through both p^*, therefore they are Steiner circles of the 1st kind.

2. For any hyperbolic triangle Δpqr, we can always find a Euclidean circle passing through p, q, r since the three points cannot be collinear. Now it is clear from the pictures that there are three possibilities, either the circle is entirely contained within $\mathbb{D}$, or it is tangent to $\partial\mathbb{D}$, or it is too big and intersect $\partial\mathbb{D}$ at two points.

3. Without loss of generality, up to rotations, we may just assume that both horocycles have ideal points at 1. In order to find a transformation T mapping a horocycle to the other, such map will necessarily fix $1 \in \partial\mathbb{D}$. This means that T is either parabolic or hyperbolic (recall from Q2 of lecture 7 practice problems). Now looking at the pictures from lecture 8, it is clear that a good candidate would be a hyperbolic transformation.

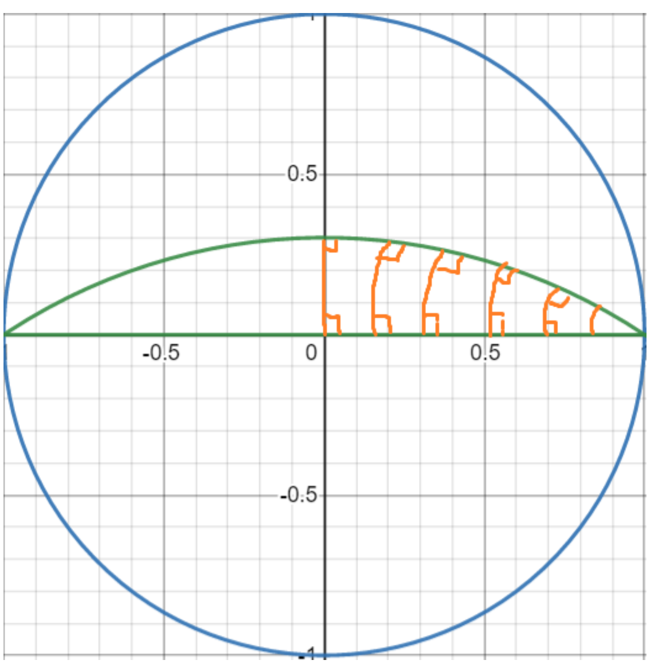
   Suppose T is a hyperbolic transformation fixing $1, -1$. It is instructive to look at the picture of its normal form. Under $S(z) = \frac{z-1}{z+1}$, we send $1 \mapsto 0$ and $-1 \mapsto \infty$. Then $S(\partial\mathbb{D})$ is just a straight line passing through 0, whereas the the image of the two horocycles under S

would be two circles that are tangent to the line at the origin. Now since $S(i) = \frac{i-1}{i+1} = i$, $S(\partial\mathbb{D})$ is the imaginary axis. We have the following picture.



Here the blue line is the image of $\partial\mathbb{D}$ whereas the green lines are images of the horocycles. Now it is clear that one can find a scaling $z \mapsto kz$ that maps one circle to another, where we can just take k to be the ratio of the radii. Such transformation is hyperbolic by definition, and the corresponding transformation $S^{-1}(kS(z))$ would be the desired transformation taking one horocycle to the other.

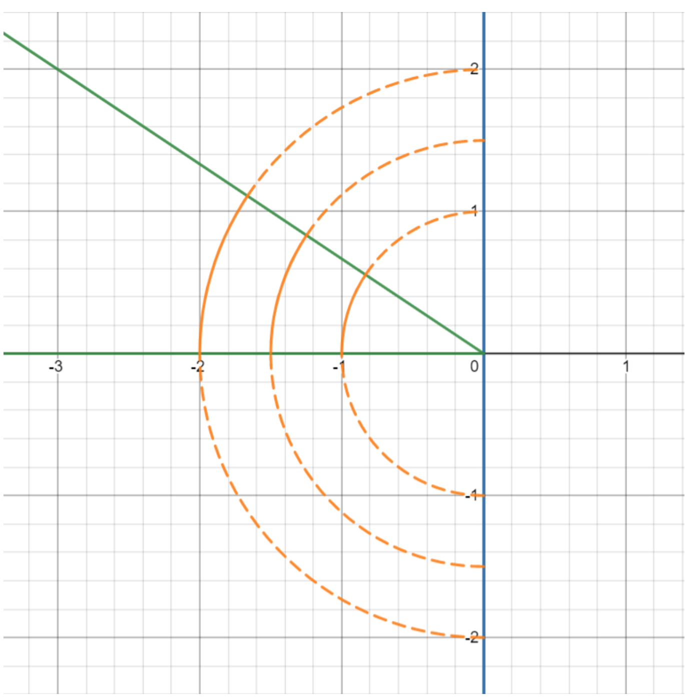
4. Without loss of generality, we can assume that the hyperbolic straight line is the real axis, by finding a transformation that takes a point on L to the origin and rotate it so that the ideal points are $1, -1$. The picture looks like this.



Here the green lines are the hyperbolic straight line L and hypercycle C. The orange lines are perpendicular between L and C, they measure the shortest distance between the two curves at particular points. Note that they are hyperbolic straight lines themselves because distances are minimized by hyperbolic straight lines. Now our goal is to find ways to show that the orange line segments all have the same lengths. The idea is that we should find transformation $T \in H$ that map one orange line segment to another. Then their distance can be seen to be the same since T preserves distance. If such a transformation exists, it is necessarily fixing $1, -1$. So again we are looking at hyperbolic transformation.

Following our general principle, we should look at the picture of the normal form. Under $S(z) = \frac{z-1}{z+1}$, mapping $1 \mapsto 0$ and $-1 \mapsto \infty$. Again $S(\partial\mathbb{D})$ is the imaginary axis, and $S(L)$ is the half ray going from 0 to ∞, passing through $S(0) = -1$, i.e. $S(L)$ is the negative x-axis. Now C is a hypercycle, in particular part of a cline, so $S(C)$ would still be a half ray going from 0 to ∞, along a different direction.

Finally, the images of the prange hyperbolic straight line segments that measure the distance would be circular arcs connecting the two half rays, where the circles are centered at the origin. The reason is that the hypercycles are Steiner circles of the 1st kind with respect to $1, -1$, meanwhile the orange hyperbolic straight lines are Steiner circles of the second kind. In particular they are actually perpendicular to all the hypercycles through $1, -1$. This means that the images of orange lines under S would be perpendicular to all the radial rays going from 0 to ∞. This implies that they are circular arcs centered at origin. The picture looks like this.



Here the horizontal green line is $S(L)$ and the slanted green line is $S(C)$. The orange lines are the images of hyperbolic lengths minimizing the distance. Again from the picture it is immediately clear that for any two orange hyperbolic straight lines, one can find a scaling $z \mapsto kz$ that takes one to another. We know that this transformation $S^{-1}(kS(z))$ preserves distance. And this completes the proof.

5. See the solution for Q2c of lecture 9 and 10 practice problems.

6. The angle C opposite to c is $\pi/2$, so by cosine rule I, we have $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \pi/2 = \cosh a \cosh b$.