THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Quiz 1 solutions
10th February 2022

1. (a) (1 point) Write $z = x + iy$ for real x, y, then breaking $z^2 = -9$ into real and imaginary parts gives $x^2 - y^2 = -9$ and $2xy = 0$. From the second equation we obtain x or y is 0. If $y = 0$, then $x^2 = -9$, which is impossible for real number x. So $x = 0$, and $y^2 = 9$. This gives $y = \pm 3$. So $z = \pm 3i$.

   (b) (1 point) Same as before, breaking $z^2 = -2i$ into real and imaginary parts gives $x^2 - y^2 = 0$ and $2xy = -2$. From the first equation, we can factorize it into $(x + y)(x - y) = 0$, so $x = y$ or $x = -y$. If $x = y$, then plugging into the second equation yields $x^2 = -1$, which is impossible for real x. So $x = -y$, and we obtain $x^2 = 1$ and hence $x = 1$ or -1. The corresponding y is -1 and 1 respectively. So $z = 1 - i$ or $-1 + i$.

   (c) (1 point) We use polar coordinates this time. $|-1-\sqrt{3}i| = 2$ so we can write $-1-\sqrt{3}i = 2e^{\frac{4\pi i}{3}}$. Then writing $z = re^{i\theta}$, we have $r^2e^{2i\theta} = 2e^{\frac{4\pi i}{3}}$. So $r = \sqrt{2}$ and $\theta = \frac{1}{2}(\frac{4\pi}{3}) = \frac{2\pi}{3}$ or $\theta = \frac{1}{2}(\frac{4\pi}{3} + 2\pi) = \frac{5\pi}{3}$. We have $z = \sqrt{2}e^{i\frac{2\pi}{3}} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$ or $\sqrt{2}e^{i\frac{5\pi}{3}} = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$.

2. (a) (1 point) We first perform a translate $z \mapsto z - (1 + i)$ to translate $1 + i$ to the origin. Then perform rotation $z \mapsto e^{i\frac{\pi}{4}}$. And translate back $z \mapsto z + (1 + i)$. Therefore $T(z) = e^{i\frac{\pi}{4}}(z - (1 + i)) + 1 + i = (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)z + 1 + (1 - \sqrt{2})i$.

   (b) (1 point) In Cartesian coordinates, if we write $a = \alpha_1 + \alpha_2 i$, $b = \beta_1 + \beta_2 i$ and $z = x+yi$, then $\text{Im}(az+b) = \alpha_1 x - \alpha_2 y + \beta_1 + (\alpha_1 y + \alpha_2 x + \beta_2)i = \alpha_1 y + \alpha_2 x + \beta 2 = 0$. This clearly defines a linear equation, which represents a straight line in the plane, as long as $a \neq 0$.

   (c) (2 points) To determine the image of $S = \{z : \text{Im}(az + b) = 0\}$ under T, one can consider the inverse function of T, which exists because T is bijective. Making z the subject in the formula for $T(z)$, we obtain

$$
\begin{aligned}
z = T^{-1}(w) &= \frac{w - 1 - (1 - \sqrt{2})i}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i} \\
&= \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) w + 1 - \sqrt{2} + i = a'w + b
\end{aligned}
$$

   So the image can be expressed as,

$$
\begin{aligned}
T(S) &= \{w = T(z) \in \mathbb{C} : \text{Im}(az + b) = 0\} \\
&= \{w \in \mathbb{C} : \text{Im}(aT^{-1}(w) + b) = 0\} \\
&= \{w \in \mathbb{C} : \text{Im}(a(a'w + b') + b) = 0\} \\
&= \{w \in \mathbb{C} : \text{Im}(aa'w + ab' + b) = 0\}
\end{aligned}
$$

   Which is again a straight line.

3. (a) (1 point) Taking $\theta = 0$ yields $R_0(z) = e^0 z = z$ is the identity map on $\mathbb{C}$.

$R_{-\theta}$ is the inverse of R_θ, because $R_\theta \circ R_{-\theta}(z) = e^{i\theta} \cdot e^{-i\theta} z = z$ and likewise for $R_{-\theta} \circ R_\theta$.

We also have $R_{\theta_1} \circ R_{\theta_2}(z) = e^{i\theta_1 + i\theta_2} z = e^{i(\theta_1+\theta_2)} z = R_{\theta_1+\theta_2}(z)$. So any compositions of the maps are again in the transformation group.

(b) (2 points) In order to check $f(z) = |z|$ is invariant. It suffices to check that $f(T(z)) = f(z)$ for any $z \in \mathbb{C}$ and $T \in G$. Check that for any z and θ,

$$
\begin{aligned}
f(R_\theta(z)) &= f(e^{i\theta} z) \\
&= |e^{i\theta} z| \\
&= |e^{i\theta}| \cdot |z| \\
&= |z| = f(z).
\end{aligned}
$$

So we are done.