THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 7 practice problems solution
3rd March 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. We just have to show any transformation written in one of the expressions can be arranged into the other expression.

   Given $T(z) = e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z}$, we can write it as

$$\begin{aligned} T(z) &= e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z} \\ &= \frac{e^{\frac{i\theta}{2}}}{e^{\frac{-i\theta}{2}}}\cdot\frac{z-z_0}{-\overline{z_0}z+1} \\ &= \frac{e^{\frac{i\theta}{2}}z - e^{\frac{i\theta}{2}}z_0}{-e^{\frac{-i\theta}{2}}\overline{z_0}z + e^{\frac{-i\theta}{2}}} \end{aligned}$$

   The last expression is in the form $\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$ where $\alpha = e^{\frac{i\theta}{2}}$ and $\beta = -e^{\frac{i\theta}{2}}z_0$, so they satisfy $|\alpha|^2 - |\beta|^2 = 1 - |z_0|^2 > 0$ since we assumed $z_0 \in \mathbb{D}$.

   Conversely, take $T(z) = \frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}$, in the following, let's write $\overline{\alpha} = \alpha^*$ to avoid confusion. We can write T as

$$\begin{aligned} T(z) &= \frac{\alpha z+\beta}{\beta^* z+\alpha^*} \\ &= \frac{\alpha}{\alpha^*}\cdot\frac{z+\frac{\beta}{\alpha}}{\frac{\beta^*}{\alpha^*}z+1} \end{aligned}$$

   This is in the form of $e^{i\theta}\frac{z-z_0}{1-\overline{z_0}z}$, if we take $z_0 = -\frac{\beta}{\alpha}$ and $e^{i\theta} = \frac{\alpha}{\alpha*}$. The latter makes sense because $\left|\frac{\alpha}{\alpha*}\right| = 1$ so it can be expressed as $e^{i\theta}$ for some angle θ.

2. (a) ($\Leftarrow$) Suppose there is one fixed point z_0 inside $\mathbb{D}$, because T preserves the unit circle, we know $T(z_0^*) = T(z_0)^*$, so its symmetric point $z_0^* = \frac{1}{\overline{z_0}}$ is also fixed. We can find another $S \in H$ that sends z_0 to 0 and sends z_0^* to $0^* = \infty$. Therefore $R(w) = STS^{-1}(w) = \lambda w$ is the normal form. Now since $S, T \in H$, by property of transformation group, we also have $R \in H$. So R also preserves the unit circle. This implies that $|\lambda z| = |z|$ so we have $|\lambda| = 1$ and T is elliptic.

   ($\Rightarrow$) This follows from ($\Leftarrow$) of part (b) in the following. Alternatively, we can prove this directly by noting that if T is elliptic, and if two fixed points lie on $\partial\mathbb{D}$, then the unit circle joining the two points is part of a Steiner circle of 1st kind, so it is not fixed by T, this contradicts the fact that T preserves the unit circle.

(b) ($\Leftarrow$) Suppose T has two fixed points z_1, z_2 on $\partial\mathbb{D}$, we can find an $S \in H$ that maps z_1 to 1 and z_2 to -1. Then STS^{-1} and T has the same normal form, so it suffices to show that $T' = STS^{-1}$ fixing $1, -1$ is hyperbolic. Consider

$$\frac{T'(z)+1}{T'(z)-1} = \lambda\frac{z+1}{z-1}$$

We can make $T'(z)$ the subject by manipulating the fractions, and obtain

$$T'(z) = \frac{(\lambda+1)z + (\lambda-1)}{(\lambda-1)z + (\lambda+1)}$$

But we know $T'(z)$ is in H, so we can rewrite it in

$$T'(z) = \frac{z + \frac{\lambda-1}{\lambda+1}}{1 + \frac{\lambda-1}{\lambda+1}z}$$

Hence we see that $\theta = 0$ and $z_0 = -\frac{\lambda-1}{\lambda+1}$ satisfies $\overline{z_0} = z_0 \in \mathbb{R}$. Now $\frac{w-1}{w+1}$ sends 0 to -1, 1 to 0 and -1 to ∞. So in particular it preserves the real line, so $\frac{\lambda-1}{\lambda+1} \in \mathbb{R}$ implies that $\lambda \in \mathbb{R}$. Now z_0 is also required to be in $\mathbb{D}$, so $-1 < \frac{\lambda-1}{\lambda+1} = 1 - \frac{2}{\lambda+1} < 1$. And so $0 < \frac{2}{\lambda+1} < 2$, or $1 < \lambda + 1 < \infty$. We obtain $\lambda > 0$. Thus we conclude that T', and hence T is hyperbolic.

($\Rightarrow$) Suppose that T is hyperbolic, it has two fixed point and by part (a) we know that it is impossible for the fixed points to lie inside $\mathbb{D}$, so the only possibility is that both fixed points lie on $\partial\mathbb{D}$. Alternatively, we can also argue that if T is hyperbolic and there is unique fixed point z in $\mathbb{D}$, then by lemma 2 of lecture 7, we know that the unit circle is a Steiner circle of the 2nd kind for z, z^*. And so by hyperbolic property, it should be moved by T, this contradicts with the fact that T preserves $\mathbb{D}$.

(c) This case is the easiest. Suppose that T has a unique fixed point, since $|z^*| = \frac{1}{|z|}$, in order to have $z^* = z$, this forces $|z| = 1$. So the unique fixed point lies on $\partial\mathbb{D}$

(d) If we have two fixed points, and one of them lies inside $\mathbb{D}$ then there will be another lying outside $\overline{\mathbb{D}}$. And if none of the two fixed points are in $\mathbb{D}$, then they have to lie on $\partial\mathbb{D}$. Otherwise there is one fixed point, and it must lie on $\partial\mathbb{D}$ as well.