THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2021-22
Lecture 7 practice problems
4th March 2022

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. Recall the subgroup of Mobius group that preserves the unit disc $\mathbb{D}$ is given by

$$H = \left\{T \in M : T(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z_0}z} \text{ for some } \theta \in \mathbb{R}, z_0 \in \mathbb{D}\right\}$$

Actually, we have another description of H given by

$$H = \left\{T(z) = \frac{\alpha z + \beta}{\overline{\beta}z + \overline{\alpha}} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 > 0\right\}$$

Check that the two descriptions are equivalent.

2. (Difficult) Recall that we classified any Mobius transformation into elliptic, hyperbolic, loxodromic and parabolic transformations. We can try to restrict our attentions to transformations $T \in H$ and see which types they fall into. We assume T is not the identity map in the following.

   (a) Prove that a transformation $T \in H$ is elliptic, i.e. T has two fixed points and it is conjugate to a rotation $z \mapsto e^{i\varphi}z$, if and only if T has a unique fixed point within $\mathbb{D}$.

   (b) Prove that a transformation $T \in H$ is hyperbolic, i.e. T has two fixed points and it is conjugate to a dilation $z \mapsto kz$ for $k > 0$, if and only if T has two distinct fixed points on $\partial\mathbb{D}$

   (c) Prove that a transformation $T \in H$ is parabolic, i.e. T has a unique fixed point, if and only if T has a unique fixed point on $\partial\mathbb{D}$.

   (d) Explain why the above three cases exhaust all possibilities, so that T is never loxodromic. In other words, any $T \in H$ is essentially the same as translation, rotation or scaling.

   Hint: It is convenient to try to transport the fixed points to some more distinguished points on $\mathbb{D}$ or $\partial\mathbb{D}$.