**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MMAT 5120 Topics in Geometry 2021-22**
**Lecture 2 practice problems**
**21st January 2022**

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **echlam@math.cuhk.edu.hk** if you have any questions.

1. Find a formula for the following rotations, as a function $f(z)$. All rotations are taken to be counter-clockwise.
   (a) Rotation by $45^\circ$ about the point $i$.
   (b) Rotation by $90^\circ$ about the point $2+i$.
   (c) Rotation by $-60^\circ$ about the point $3$.

   Hint: We know how to rotate at the origin $0$. Try to turn the given points into the "origin".

2. Invert the stereographic projection, i.e., express $a, b, c$ in terms of $x, y$ in the following equation:
$$x + iy = \frac{a+ib}{1-c}.$$

3. In the stereographic projection, we define a map $S : \mathbb{S}^2 \setminus \{(0,0,1)\} \to \mathbb{C}$ from the sphere with north pole removed to $\mathbb{C}$ by drawing a straightline from the sphere to north pole. We can mimick this construction, instead of using the north pole, we pick the south pole and let $S' : \mathbb{S}^2 \setminus \{(0,0,-1)\} \to \mathbb{C}$ be the projection.
   (a) Show that the formula of $S'(a,b,c) = x + iy$ is given by $S'(a,b,c) = \frac{a+ib}{1+c}$. (Hint: follow the idea of lecture 2 page 10.)
   (b) Let's fix a point $(a,b,c) \in \mathbb{S}^2$ that is neither the north nor the south pole, show that $S(a,b,c) \cdot \overline{S'(a,b,c)}$ is always equal to 1. (The bar means conjugate.)
   (c) Given $a, b, c$, find $a', b', c'$ so that $S(a,b,c) = S'(a',b',c')$. (Hint: Instead of directly computing, think in terms of the symmetry of spheres.)
   (d) Explain why the $180^\circ$ rotation of the sphere along the x-axis, i.e. $g : (a,b,c) \mapsto (a,-b,-c)$ is a lift of the inversion $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by $z \mapsto \frac{1}{z}$, where we take $f(0) = \infty$ and $f(\infty) = 0$. (Hint: if you reflect the sphere along the $xy$-plane and then the $xz$-plane, its the same as rotation along $x$-axis by $180^\circ$.)

   Remark: In less fancy terms, $f, g$ represents the same functions because the stereographic projection is a bijective map. So all this exercise is saying that, the inverse $f(z) = \frac{1}{z}$ on the complex plane can be understood as a simple rotation of $\mathbb{S}^2$.