MATH 2060 Mathematical Analysis II
Tutorial Class 9
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1. (a) Prove that if $\{f_n\}$ be a sequence of Riemann integrable function on $[a, b]$ and f_n converge uniformly to f on $[a, b]$, then $f \in R[a, b]$ and $\int_a^b f = \lim_n \int_a^b f_n$.

   (b) Let $\{f_n\}$ be a sequence of functions that converges uniformly to f on A and that satisfies $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in A$. If g is continuous on $[-M, M]$, show that $\{g \circ f_n\}$ converges uniformly to $g \circ f$ on A.

2. Given an example of sequence of Riemann integrable functions $\{f_n\}$ on $[0, 1]$ converging pointwisely to f on $[0, 1]$ such that

   (a) $f \in R[0, 1]$ but $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

   (b) f is bounded but f is not Riemann integrable on $[0, 1]$.

3. Give an example of sequence of functions (fn) on $[0, 1]$ satisfying

   (a) for all n, f_n is discontinuous at any point of $[0, 1]$, but f_n converge uniformly to a continuous function f on $[0, 1]$.

   (b) $\{f_n\}$ converge pointwisely to f on $[0, 1]$ but the convergence is not uniform on any subinterval of $[0, 1]$.

4. (a) State the Bounded Convergence Theorem.

   (b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Give a sequence of continuous function $\{g_n\}$ on $[a, b]$ such that $|g_n| \leq 1$ on $[a, b]$ and $\{fg_n\}$ converge pointwisely to $|f|$ on $[a, b]$.

   (c) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose $\int_a^b fg \leq 1$ for all continuous function g on $[a, b]$, prove that $\int_a^b |f| \leq 1$.

5. Let $f_n \in C^1([a, b]), n \in \mathbb{N}$. Show that if f_n' converge uniformly to some function φ on $[a, b]$ and there exists a point $x_0 \in [a, b]$ for which $\{f_n(x_0)\}$ converges, then the sequence of functions $\{f_n\}$ converges uniformly on $[a, b]$ to some function $f \in C^1([a, b])$ and f_n' converges uniformly to $f' = \varphi$.