THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 5: Section 7.2: 8, 9, 12

8. Suppose that f is continuous on $[a, b]$, that $f(x) \geq 0$ for all $x \in [a, b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (4 marks)

Solution. Suppose $f(c) > 0$ for some $c \in (a, b)$. By continuity of f, there is some $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq (a, b)$ and $f(x) > \frac{f(c)}{2}$ for every $x \in (c - \delta, c + \delta)$. Therefore,

$$\int_{c-\delta}^{c+\delta} f \geq \frac{f(c)}{2}(2\delta) > 0.$$

Since $f \geq 0$, we have $\int_a^b f \geq \int_{c-\delta}^{c+\delta} f > 0$, which is a contradiction.

This shows that $f|_{(a,b)} \equiv 0$, and by continuity of f, $f \equiv 0$ on $[a, b]$.

9. Show that the continuity hypothesis in the preceding exercise cannot be dropped. (2 marks)

Solution. Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } 0 < x \leq 1. \end{cases}$$

Clearly, $f \geq 0$ and $\int_0^1 f = 0$, but $f \not\equiv 0$ and f is discontinuous at 0.

12. Show that $g(x) := \sin(1/x)$ for $x \in (0, 1]$ and $g(0) := 0$ belongs to $\mathcal{R}[0, 1]$. (4 marks)

Solution. Let $\epsilon > 0$. It suffices to propose a partition $\mathcal{P}$ on $[0, 1]$ so that

$$\mathcal{U}(g, \mathcal{P}) - \mathcal{L}(g, \mathcal{P}) < \epsilon.$$

Let $\eta > 0$. Since $g|_{[\eta,1]}$ is continuous, g is uniform continuous on $[\eta, 1]$. There is some $\delta > 0$ such that

$$|g(x) - g(y)| < \epsilon/2 \quad \text{for } x, y \in [\eta, 1] \text{ with } |x - y| < \delta.$$

Let $\mathcal{P}_1$ be a partition on $[\eta, 1]$ with $\|\mathcal{P}_1\| < \delta$, and let $\mathcal{P} = \{0\} \cup \mathcal{P}_1$. We may write $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$, where

$$0 = x_0 < x_1 = \eta < x_2 < \ldots < x_n = 1.$$

We denote $I_i = [x_{i-1}, x_i]$. Notice that

$$\begin{aligned} \mathcal{U}(g, \mathcal{P}) - \mathcal{L}(g, \mathcal{P}) &= \sum_{i=1}^{n} \sup_{x,y \in I_i} |g(x) - g(y)| \cdot |I_i| \\ &= \sup_{0 \leq x,y \leq \eta} |g(x) - g(y)| \cdot \eta + \sum_{i=2}^{n} \sup_{x,y \in I_i} |g(x) - g(y)| \cdot |I_i| \end{aligned}$$

The function $|g|$ is bounded by 1, so $\sup_{0\le x,y\le\eta}|g(x)-g(y)|\le 2$. Uniform continuity of g on $[\eta,1]$ together with $|I_i|<\delta$, yield that $\sup_{x,y\in I_i}|g(x)-g(y)|\le\epsilon/2$ for $i\ge 2$. Therefore,

$$\mathcal{U}(g,\mathcal{P})-\mathcal{L}(g,\mathcal{P})\le 2\eta+\frac{\epsilon}{2}(1-\eta)\le 2\eta+\frac{\epsilon}{2}$$

If we choose $\eta<\epsilon/4$ at the beginning, we will obtain a partition $\mathcal{P}$ on $[0,1]$ so that

$$\mathcal{U}(g,\mathcal{P})-\mathcal{L}(g,\mathcal{P})<\epsilon.$$

This completes the proof.