THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2020)
Suggested Solution of Homework 4: Section 6.4: 4, 9, 10

4. Show that if $x > 0$, then $1 + \frac{1}{2}x - \frac{1}{8}x^2 \le \sqrt{1+x} \le 1 + \frac{1}{2}x$. (3 marks)

Solution. Let $f(x) = \sqrt{1+x}$. Then,

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{1+x}}, & f'(0) &= \frac{1}{2}; \\ f''(x) &= -\frac{1}{4(1+x)^{\frac{3}{2}}}, & f''(0) &= -\frac{1}{4}; \\ f'''(x) &= \frac{3}{8(1+x)^{\frac{5}{2}}}. \end{aligned}$$

By Taylor's Theorem, there is $c_1 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(c_1)}{2}(x-0)^2 \\ &= 1 + \frac{1}{2}x - \frac{1}{8(1+c_1)^{\frac{3}{2}}}x^2 \qquad (1) \end{aligned}$$

Similarly, there is $c_2 \in (0, x)$ such that

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f''(c_2)}{3!}(x-0)^3 \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16(1+c_2)^{\frac{5}{2}}}x^3 \qquad (2) \end{aligned}$$

Equation (1) gives $\sqrt{1+x} \le 1 + \frac{1}{2}x$, while Equation (2) gives $1 + \frac{1}{2}x - \frac{1}{8}x^2 \le \sqrt{1+x}$. This completes the proof.

9. If $g(x) := \sin x$, show that the remainder term in Taylor's Theorem converges to zero as $n \to \infty$ for each fixed x_0 and x. (3 marks)

Solution. Recall that the remainder term for the nth Taylor's polynomial is given by

$$R_n(x) = \frac{g^{(n+1)}(c_n)}{(n+1)!}(x-x_0)^{n+1} \quad \text{for some } c_n \text{ between } x \text{ and } x_0.$$

Notice that $g^{(n+1)}(x) = \sin x, -\sin x, \cos x$ or $-\cos x$. Hence, $|g^{(n+1)}(c_n)| \le 1$. Therefore, we have

$$|R_n(x)| \le \frac{|x-x_0|^{n+1}}{(n+1)!}.$$

Let $a_n = \frac{|x-x_0|^{n+1}}{(n+1)!}$. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{|x-x_0|}{n+2} = 0 < 1$, ratio test tells us that $\lim_{n\to\infty} a_n = 0$. Therefore, $\lim_{n\to\infty} |R_n(x)| = 0$ by sandwich theorem.

10. Let $h(x) := e^{-1/x^2}$ for $x \neq 0$ and $h(0) := 0$. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that the remainder term in Taylor's Theorem for $x_0 = 0$ does *not* converge to zero as $n \to \infty$ for $x \neq 0$. [*Hint:* By L'Hospital's Rule, $\lim_{x \to 0} h(x)/x^k = 0$ for any $k \in \mathbb{N}$. Use Exercise 3 to calculate $h^{(n)}(x)$ for $x \neq 0$.] (4 marks)

Solution. First, we show that $\lim_{x \to 0} h(x)/x^k = 0$ for any $k \in \mathbb{N}$. Note that

$$\lim_{x \to 0} h(x)/x^k = \lim_{x \to 0} \frac{e^{-1/x^2}}{x^k} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x^{2k}} x^k$$

Let $y = 1/x^2$. As $x \to 0$, $y \to \infty$. We have

$$\lim_{x \to 0} \frac{e^{-1/x^2}}{x^{2k}} = \lim_{y \to \infty} \frac{y^k}{e^y} = 0.$$

The last equality is due to a successive application of L'Hospital's Rule. This shows that $\lim_{x \to 0} h(x)/x^k = 0$

Second, we calculate $h^{(n)}(x)$ for $x \neq 0$. Notice that

$$h'(x) = \frac{2}{x^3} e^{-1/x^2} = \frac{2}{x^3} h(x).$$

From the formula above, if h is n-times differentiable, then h' is also n-times differentiable, and hence h is (n+1)-times differentiable. Inductively, we see that h is infinitely differentiable.

We apply Leibniz's rule to find $h^{(n+1)}(x)$ for $x \neq 0$. By formula above, we have

$$h^{(n+1)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{x^3}\right)^{(n-k)} h^{(k)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{(n-k+2)!}{x^{n-k+3}} h^{(k)}(x).$$

Third, we do induction on $n \in \mathbb{N}$ to argue the following.

(i) $\lim_{x \to 0} \frac{h^{(n)}(x)}{x^k} = 0$ for all $k \in \mathbb{N}$

(ii) $h^{(n)}(0) = 0$

For the case $n = 1$, by first part of our solution, we have

$$\lim_{x \to 0} \frac{h'(x)}{x^k} = \lim_{x \to 0} \frac{2h(x)}{x^{3+k}} = 0.$$

This verifies (i). On the other hand, we can verify (ii) by the same argument:

$$h'(0) = \lim_{x \to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0} \frac{h(x)}{x} = 0.$$

Assume both conditions (i), (ii) hold for $n = 1, 2, \ldots, N$. We check that these conditions also hold for $n = N + 1$. To see this, by Leibniz's rule and induction hypothesis,

$$\lim_{x\to 0} \frac{h^{(N+1)}(x)}{x^k} = \sum_{j=0}^{N} \binom{N}{j} (-1)^{N-j}(N-j+2)! \left(\lim_{x\to 0} \frac{h^{(j)}(x)}{x^{N-j+3+k}} \right) = 0$$

Moreover,

$$h^{(N+1)}(0) = \lim_{x\to 0} \frac{h^{(N)}(x) - h^{(N)}(0)}{x - 0} = \lim_{x\to 0} \frac{h^{(N)}(x)}{x} = 0.$$

This completes the induction.

Finally, note that the remainder term for the nth Taylor's polynomial is

$$R_n(x) = h(x) - \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} x^k = h(x)$$

Therefore, $\{R_n(x)\}_{n=1}^{\infty}$ is a nonzero constant sequence whenever $x \neq 0$ is fixed. The limit is nonzero.