**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH2060B Mathematical Analysis II (Spring 2020)**
**Suggested Solution of Homework 1: Section 5.4: 4, 6, 7**

4. Show that the function $f(x) := 1/(1+x^2)$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$. (2 marks)

**Solution.**

**Method 1.** Since $f$ is a continuous function on $\mathbb{R}$ such that $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ exist, $f$ is uniformly continuous on $\mathbb{R}$. We may show why this is true. Let $\epsilon > 0$. By Cauchy criteria, there is $M > 0$ such that

(a) if $x, y > M$, then $|f(x) - f(y)| < \epsilon$
(b) if $x, y < -M$, then $|f(x) - f(y)| < \epsilon$

Moreover, since $f$ is uniformly continuous on the closed and bounded interval $[-M-1, M+1]$, there is $\delta > 0$ such that if $x, y \in [-M-1, M+1]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Fix $\delta' = \min(\delta, 1)$, if $x, y \in \mathbb{R}$ and $|x - y| < \delta'$, then we can conclude that one of the three cases must occur:

(i) $x, y > M$
(ii) $x, y < -M$
(iii) $x, y \in [-M-1, M+1]$ and $|x - y| < \delta$

Therefore, $|f(x) - f(y)| < \epsilon$. This completes the proof.

**Method 2.**

$$\begin{aligned} f(x) - f(y) &= \frac{1}{1+x^2} - \frac{1}{1+y^2} \\ &= \frac{y^2 - x^2}{(1+x^2)(1+y^2)} \\ &= \frac{(y-x)(y+x)}{(1+x^2)(1+y^2)} \end{aligned}$$

Note that $\left|\dfrac{x}{1+x^2}\right| \le \frac{1}{2}$ iff $2|x| \le 1 + x^2$ iff $(|x|-1)^2 \ge 0$.
Therefore, $\left|\dfrac{y+x}{(1+x^2)(1+y^2)}\right| \le \dfrac{|x|}{1+x^2} + \dfrac{|y|}{1+y^2} \le 1$, hence $|f(x) - f(y)| \le |y - x|$. $f$ is a Lipschitz function, and hence uniformly continuous.

6. Show that if $f$ and $g$ are uniformly continuous on $A \subseteq \mathbb{R}$ and if they are both bounded on $A$, then their product $fg$ is uniformly continuous on $A$. (2 marks)

**Solution.**

Let $M > 0$ such that $|f(x)|, |g(x)| \leq M$ for all $x \in A$. Let $\epsilon > 0$. By uniform continuity of $f, g$, there are $\delta_1, \delta_2 > 0$ such that

(i) if $x, y \in A$ and $|x - y| < \delta_1$, $|f(x) - f(y)| < \frac{\epsilon}{2M}$

(ii) if $x, y \in A$ and $|x - y| < \delta_2$, $|g(x) - g(y)| < \frac{\epsilon}{2M}$

Let $\delta = \min(\delta_1, \delta_2)$. If $x, y \in A$ and $|x - y| < \delta$, then

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)| \\ &= |g(x)|\,|f(x) - f(y)| + |f(y)|\,|g(x) - g(y)| \\ &< M\,\frac{\epsilon}{2M} + M\,\frac{\epsilon}{2M} = \epsilon \end{aligned}$$

Therefore, $fg$ is uniformly continuous on $A$.

7. If $f(x) := x$ and $g(x) := \sin x$, show that both $f$ and $g$ are uniformly continuous on $\mathbb{R}$, but their product $fg$ is not uniformly continuous on $\mathbb{R}$. (6 marks)

**Solution.**

We may show that $f, g$ are Lipschitz functions. For function $g$,

$$\begin{aligned} |\sin x - \sin y| &= |2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})| \\ &\leq 2 \cdot 1 \cdot |\sin(\frac{x-y}{2})| \\ &\leq 2 \cdot |\frac{x-y}{2}| = |x - y| \end{aligned}$$

To show that $fg$ is not uniformly continuous on $\mathbb{R}$, we may argue that for any $\delta > 0$, there is some $x \in \mathbb{R}$ such that $|fg(x+\delta) - fg(x)| \geq 1$. Indeed,

$$\begin{aligned} |fg(x+\delta) - fg(x)| &= |(x+\delta)\sin(x+\delta) - x\sin x| \\ &\geq |x(\sin(x+\delta) - \sin x)| - \delta \\ &= |x|\,|2\cos(x + \frac{\delta}{2})\sin(\frac{\delta}{2})| - \delta \end{aligned}$$

We may assume $\delta < 1$, so that $\sin(\frac{\delta}{2}) > 0$. For such fixed $\delta$, we choose $x = 2\pi N - \frac{\delta}{2}$ for some large $N \in \mathbb{N}$. Therefore, $|fg(x+\delta) - fg(x)| \geq (4\pi N - \delta)\sin(\frac{\delta}{2}) - \delta$. Clearly, when $N$ is chosen properly according to $\delta$, we will have $|fg(x+\delta) - fg(x)| \geq 1$.