Suppose that

$$e = \frac{n_0}{m_0}$$

for some $n_0, m_0 \in \mathbb{N}$, $m_0 \neq 0$

$$0 < \frac{n_0}{m_0} - \left(1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}\right) < \frac{3}{(n+1)!}$$

$\forall n = 1, 2 \ldots$

$$\Rightarrow 0 < \frac{n_0 n!}{m_0} - \underbrace{n!\left(1+1+\dots+\frac{1}{n!}\right)}_{\in \mathbb{N}} < \frac{3 n!}{(n+1)!} = \frac{3}{n+1} < 1$$

$\in \mathbb{N}$

Choose $n \gg$ s.t.

$$\frac{n_0 n!}{m_0} \in \mathbb{N}.$$

Contradiction! $\square$

Prop: Let $f:(a,b)\to\mathbb{R}$ be a funct

Let $x_0\in(a,b)$

Then

(i) If $f'(x_0)$ exists, then

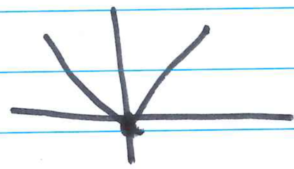
$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}.$$

↑

(symmetic derivatives)

(Remark): Consider $f(x)=|x|$, $x\in(-1,1)$

N.B: $f'(0)$ does not exist



$$\lim_{h\to 0}\frac{f(0+h)-f(0-h)}{2h}=0$$)

(ii) If $f''(x_0)$ exists and $f'(x)$ exists on (a,b)

then $$f''(x_0)=\lim_{h\to 0}\frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$$

Pf (1-(i)):

$$f'(x_0) \overset{(*)}{=} \lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$$

Pf (*):

~~NB~~ $\cancel{f'(x_0) = \lim_{h\to 0+} \frac{f(x_0+h)-f(x_0-h)}{2h}}$

$\cancel{\lim_{h\to 0-} \frac{f(x_0+h)-f(x_0-h)}{2h}}$

$\because$ $f'(x_0)$ exists

$$\therefore \lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h}$$

$$= f'(x_0)$$

$$\therefore 2f'(x_0) = \lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h} + \lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h}$$

N.B. Put $h = -t$ into $\dfrac{f(x_0+h)-f(x_0)}{h}$

$$\Rightarrow \lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h} =$$

$$\lim_{t\to 0+} \frac{f(x_0-t)-f(x_0)}{-t}$$

$\therefore 2f'(x_0) =$

$$\lim_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{t \to 0+} \frac{f(x_0-t) - f(x_0)}{-t}$$

$$= \lim_{h \to 0+} \frac{f(x_0+h) - f(x_0-h)}{h}$$

$\square$

(ii) P: If $f''(x)$ exists $\Rightarrow$

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2}$$

pf: Consider

$$g(h) = f(x_0+h) + f(x_0-h) - 2f(x_0)$$

N.B: $g(0) = 0$

L'hospital rule

$$\lim_{h\to 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$$

$$= \lim_{h\to 0} \frac{f'(x_0+h)-f'(x_0-h)}{2h}$$

$$\overset{\text{by}}{\underset{(i)}{=}} \quad f''(x_0)$$

$\square$

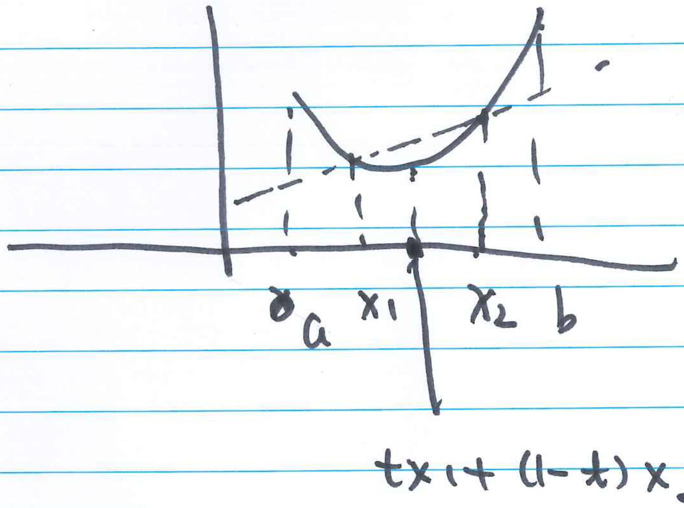
Def: let $I = [a, b]$

let $f : [a, b] \to \mathbb{R}$ be a ~~convex~~ function.

We call f a "convex" function if

$$f(t x_1 + (1-t) x_2) \leq t f(x_1) + (1-t) f(x_2)$$

$$\forall \ a \leq x_1 < x_2 \leq b, \quad \forall \ t \in [0,1]$$



Prop: Assume $f \in C^2(a,b)$. Then

f is convex iff $f''(x) \geq 0,\ \forall x \in (a,b)$

pf: "$\Rightarrow$" Assume f is ~~convex~~ convex

Fix $x_0 \in (a,b)$

⊢: $f''(x_0) \geq 0$

(pf ⊢):

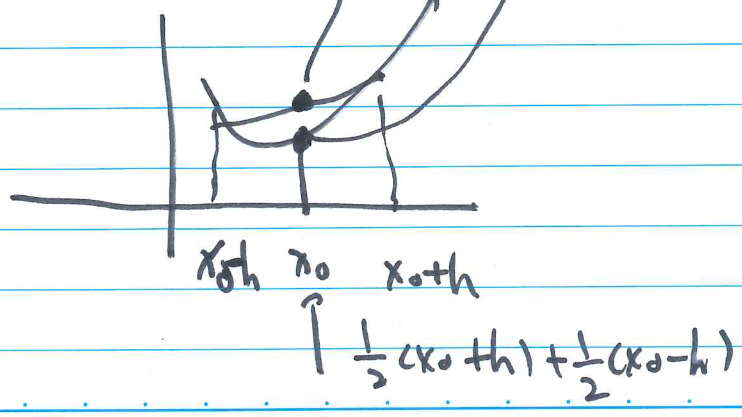
$$f''(x_0) \overset{(*)}{=} \lim_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2}$$

Note: Since f is convex,

$$\left(x_0 = \tfrac{1}{2}(x_0+h) + \tfrac{1}{2}(x_0-h)\right)$$

$$\tfrac{1}{2} f(x_0+h) + \tfrac{1}{2} f(x_0-h) \geq f(x_0)$$

$\therefore$ $f''(x_0) \geq 0$



"$\Leftarrow$" Assume $\boxed{f''(x) \geq 0}$ $\forall x$

Fix $a < x_1 < x_2 < b$

Let $x_0 = t x_1 + (1-t) x_2, \quad 0 \leq t \leq 1$

$$\boxed{?\ \ t f(x_1) + (1-t) f(x_2) \overset{?}{\geq} f(x_0)\ \ ?}$$

Note

$$f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + \frac{1}{2!} f''(c_1)(x_1 - x_0)^2$$

for some $c_1 \in (x_0, x_1)$

$$f(x_2) = f(x_0) + f'(x_0)(x_2 - x_0) + \frac{1}{2!} f''(c_2)(x_2 - x_0)^2$$

for some c_2 between x_0 and x_2

$$\Rightarrow \quad t f(x_1) + (1-t) f(x_2) =$$

$$f(x_0) + f'(x_0)\left(\cancelto{0}{t x_1 + (1-t) x_2 - x_0}\right) +$$

$$+ t \frac{1}{2!} \underbrace{f''(c_1)}_{\geq 0} \underbrace{(x_1 - x_0)^2}_{\geq 0} + (1-t) \frac{1}{2!} \underbrace{f''(c_2)}_{\geq 0} \underbrace{(x - x_0)^2}_{\geq 0}$$

$$\geq f(x_0) \qquad \square$$