MMAT5010 2021 Home Test 1

Q1. (i) Let $T : (X, ||\cdot||_1) \to (X, ||\cdot||_\infty)$ be defined by $Tf(x) = \int_a^x f(t)\,dt$. Then

$$\begin{aligned} ||Tf||_\infty &= \sup_{x\in[a,b]} |Tf(x)| \\ &\le \sup_{x\in[a,b]} \int_a^x |f(t)|\,dt \\ &\le \int_a^b |f(t)|\,dt = ||f||_1 \end{aligned}$$

Therefore $||T|| \le 1$. Furthermore, if we let $f : [a,b] \in \mathbb{R}$ to be $f(x) \equiv \frac{1}{b-a}$, then $||f||_1 = 1$ and

$$Tf(x) = \frac{x-a}{b-a}$$

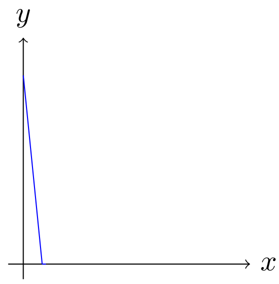
We have $||Tf||_\infty = 1$. Hence $||T|| = 1$.

(ii) Let $T : (X, ||\cdot||_1) \to (X, ||\cdot||_1)$ be defined by $Tf(x) = \int_a^x f(t)\,dt$. Then

$$\begin{aligned} ||Tf||_1 &= \int_a^b |Tf(t)|\,dt \\ &\le \int_a^b \int_a^t |f(s)|\,ds\,dt \\ &\le \int_a^b \int_a^b |f(s)|\,ds\,dt \\ &= (b-a)||f||_1 \end{aligned}$$

Therefore $||T|| \le b-a$. We claim that $||T|| = b-a$ by finding a sequence (f_n) in X with $||f_n||_1 = 1$ and $||Tf_n||_1 \to b-a$. Our f_n is defined by the followings:

- $f_n = 0$ on $[a+\frac{1}{n}, b]$
- $f_n(a) = 2n$
- f_n is a straight line on $[a+\frac{1}{n}, b]$



graph of f_n

It is easy to check that $||f_n||_1 = 1$ and $Tf_n(x) = 1$ on for $x \in [a+\frac{1}{n}, b]$. Thus $||Tf_n||_1 \ge b-(a+\frac{1}{n})$ for every n. Hence f_n is the desired sequence and $||T|| = b-a$.

Q2. (a) Let $M \subset X$ be a closed subspace, $\pi : X \to X/M$ be the canonical quotient map. Let B_X, $B_{X/M}$ be the **open** unit ball of X and X/M respectively. We claim that $\pi(B_X) = B_{X/M}$.

Suppose $x \in B_X$, then it is clear that $\pi(x) \in X_{X/M}$ because $||\pi|| \le 1$. Suppose $\bar{x} \in B_{X/M}$. First, there exists some $x \in X$ such that $\pi(x) = \bar{x}$. Because $||\pi(x)|| < 1$, there exists some $m \in M$ such that $||x - m|| < 1$. So $x - m \in B_X$ and $\pi(x-m) = \bar{x}$. It follows that

$$||\bar{F}|| = \sup_{\bar{x}\in B_{X/M}} \bar{F}(x) = \sup_{x\in B_X} \bar{F}(\pi(x)) = ||\bar{F}\circ\pi||$$

(b) Let $a \notin M$. By the Hahn-Banach theorem, there exists $\bar{F} \in (X/M)^*$, $||\bar{F}|| = 1$, $\bar{F}(\pi(a)) = ||\pi(a)|| = d(a, M)$. The desired $f \in X^*$ is given by

$$f(x) = \frac{1}{||\pi(a)||}\bar{F}(\pi(x))$$

Q3. (i) Fix $x \in c_0$. Let $y \in \ell_1$. To show that $M_x(y) \in \ell_1$, we must show that

$$\sum_{k=1}^{\infty} |x(k)y(k)| < \infty$$

Observe that for each $N = 1, 2, \dots,$

$$\sum_{k=1}^{N} |x(k)y(k)| \le \Big(\sup_{j \in \mathbb{N}} |x(j)|\Big) \sum_{k=1}^{N} |y(k)| \le ||x||_\infty ||y||_1$$

Therefore $\sum_{k=1}^{\infty} |x(k)y(k)| < \infty$. Hence M_x is well-defined.

(ii) In (i) we show that $||M_x(y)||_1 \le ||x||_\infty ||y||_1$, i.e. $||M_x|| \le ||x||_\infty$. Let $e_k = (0, 0, \dots, 0, 1, 0, \dots)$ be the canonical basis vectors in ℓ_1. We have $||e_k|| = 1$ for all k and $||M_x(e_k)|| = |x(k)|$. So $||M_x|| \ge |x(k)|$ for all k and Hence $||M_x|| \ge ||x||_\infty$.

(iii) The adjoint operator $M_x^* : \ell_\infty \to \ell_\infty$ satisfies

$$M_x^*(\xi)(y) = \xi(M_x y)$$

for all $\xi \in \ell_\infty$ and for all $y \in \ell_1$. Not $\xi(M_x y)$ is just a number in $\mathbb{R}$ (or $\mathbb{C}$):

$$\xi(M_x y) = \xi(1)x(1)y(1) + \xi(2)x(2)y(2) + \dots$$

We see that $M_x^*\xi = (x(1)\xi(1), x(2)\xi(2), \dots) \in \ell_\infty$.

Q4. Let $x \in X$. By the Hahn-Banach theorem there exists $f \in B_{X^*}$ such that $f(x) = ||x||$. Since $B_{X^*} \subset \cup_{k=1}^{n} B(x_k^*, r)$, there exists k_0 such that $f \in B(x_{k_0}^*, r)$. Then

$$\begin{aligned} ||Tx||_\infty &= \sup_k |x_k^*(x)| \\ &\ge |x_{k_0}^*(x)| \\ &\ge |f(x)| - |x_{k_0}^*(x) - f(x)| \\ &\ge ||x|| - ||x_{k_0}^* - f||\,||x|| \\ &\ge (1 - r)||x|| \end{aligned}$$