MMAT5010 2021 Assignment 8

Q1. Let $\epsilon > 0$. Let us see what happens when $||x|| = ||y|| = 1$ and $||\frac{x+y}{2}|| > 1 - \delta$. First,

$$\begin{aligned} ||\frac{x+y}{2}||^2 &= \langle \frac{x+y}{2}, \frac{x+y}{2} \rangle \\ &= \frac{1}{4}(2 + 2\langle x, y \rangle) \\ &> (1-\delta)^2 \end{aligned}$$

Thus $\langle x, y \rangle > 2(1-\delta)^2 - 1$. Then $||x - y||^2 = 2 - 2\langle x, y \rangle < 4 - 4(1-\delta)^2$. This suggests that, for every $\epsilon > 0$, we must choose $\delta > 0$ very small such that $0 \leq 4 - 4(1-\delta)^2 < \epsilon$. And we can always choose such δ, this is because $0 < 4 - 4(1-\delta)^2$ for $\delta \in (0, 1)$ and

$$\lim_{\delta \to 1-} 4 - 4(1-\delta)^2 = 0$$

This says that for every $\epsilon > 0$, there exists $\delta \in (0, 1)$ such that $0 < 4 - 4(1-\delta)^2 < \epsilon$.

Q2. ($\Rightarrow$) This is corollary 9.5 of Lecture notes.

($\Leftarrow$) It suffices to show that $V^\perp$ is closed for every subspace $V \subset X$. (Then we can replace V by $M^\perp$)

Let x_n be a sequence in $V^\perp$, $x_n \to x \in X$, we must show that $x \in V^\perp$. Let $v \in V$, by definition

$$\langle x_n, v \rangle = 0$$

Since $\langle \cdot, v \rangle$ is continuous, take $n \to \infty$ gives $\langle x, v \rangle = 0$.