MMAT5010 2021 Assignment 3

Q1. Because all norms are equivalent in finite dimensional spaces, it only needs to show A is a bounded linear operator from $(\mathbb{K}^n, ||\cdot||_\infty)$ to itself, here $||\cdot||_\infty$ is the supremum norm.

For any $x \in \mathbb{K}^n$, write $x = (x_1, x_2, \dots, x_n)$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (the i-th entry is 1, others are 0). Then

$$||Ax||_\infty = ||A(\sum_{i=1}^{n} x_i e_i)||_\infty \leq |x_1|\,||Ae_1|| + \cdots + |x_n|\,||Ae_n|| \leq \left(n \max_{i=1,\dots n} ||Ae_i||\right) \max_{i=1,\dots n} |x_i|$$

Hence A is a bounded linear operator with $||A|| \leq n \max_{i=1,\dots n} ||Ae_i||$.

Q2. Suppose (x_n) is a sequence in ℓ_1, $x_n \to x \in \ell_1$ in $||\cdot||_1$-norm. Then $x_n \to x$ in $||\cdot||_\infty$-norm because $||\cdot||_\infty \leq ||\cdot||_1$, and therefore

$$||x_n - x||_\infty \leq ||x - x_n||_1 \to 0 \text{ as } n \to \infty$$

The converse statement is: suppose (x_n) is a sequence in ℓ_1, $x_n \to x \in \ell_1$ in $||\cdot||_\infty$-norm, then $x_n \to x$ in $||\cdot||_1$-norm.

This statement is disproved by finding (x_n), x, $x_n \to x$ in $||\cdot||_\infty$ but $x_n \nrightarrow x$ in $||\cdot||_1$. Define

- $x_1 = (1, 0, 0, \dots) \in \ell_1$
- $x_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0 \dots) \in \ell_1$
- $x_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots) \in \ell_1$
- $\dots$

Then $||x_n - 0||_\infty = \frac{1}{n} \to 0$, but $||x_n - 0||_1 = 1$ for all n.