# Suggested Solution to Homework 5

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**P140, 7.** Show that in an inner product space, $x \perp y$ if and only if we have $\|x+\alpha y\| = \|x-\alpha y\|$ for all scalars $\alpha$.

**Proof.** It follows from the definition of inner product that

$$\|x+\alpha y\|^2 = \langle x+\alpha y, x+\alpha y\rangle = \langle x,x\rangle + \bar{\alpha}\langle x,y\rangle + \alpha\langle y,x\rangle + \alpha\bar{\alpha}\langle y,y\rangle = \|x\|^2 + 2Re(\bar{\alpha}\langle x,y\rangle) + |\alpha|^2\|y\|^2$$

Similarly, replacing $\alpha$ by $-\alpha$ above, one has

$$\|x-\alpha y\|^2 = \|x\|^2 - 2Re(\bar{\alpha}\langle x,y\rangle) + |\alpha|^2\|y\|^2$$

Then, $\|x+\alpha y\| = \|x-\alpha y\|$ for all scalars $\alpha$ if and only if $Re(\bar{\alpha}\langle x,y\rangle) = 0$ for all $\alpha$. Taking $\alpha = 1$ and $\alpha = i$ respectively, we conclude that $\langle x,y\rangle = 0$, i.e. $x \perp y$. □

**P167, 7.** Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$. Show that for every $x \in H$, the vector

$$y = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$$

exists in $H$ and $x - y$ is orthogonal to every $e_k$.

**Proof.** From the Bessel inequality in Theorem 3.4-6, we see that, for every $x \in H$, the series

$$\sum_{k=1}^{\infty}|\langle x, e_k\rangle|^2 \le \|x\|^2$$

converges. So, $y = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$ exists in $H$. Furthermore,

$$\langle x-y, e_j\rangle = \langle x, e_j\rangle - \langle\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k, e_j\rangle = 0$$

Hence, $x - y$ is orthogonal to $e_k$. □

**P167, 8.** Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$, and let $M = \text{span}(e_k)$. Show that for any $x \in H$ we have $x \in \bar{M}$ if and only if $x$ can be represented by $x = \sum_{k=1}^{\infty}\alpha_k e_k$ with coefficients $\alpha_k = \langle x, e_k\rangle$.

**Proof.** Assume that $x$ can be represented by $x = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$. Since $x_n = \sum_{k=1}^{n}\langle x, e_k\rangle e_k \in M$ and $x_n \to x$ as $n \to +\infty$, we have $x \in \bar{M}$. On the other hand, assume $x \in \bar{M}$. Set $y = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$. It follows from Q7 above that $x - y \perp e_k$. By the continuity of inner product, we have $x - y \perp \bar{M}$. It is clear that $x - y \in \bar{M}$. So, $x - y \in \bar{M} \cap \bar{M}^{\perp} = \{0\}$. That is, $x = y = \sum_{k=1}^{\infty}\langle x, e_k\rangle e_k$. □

**P175, 4.** Derive from (3) the following formula (which is often called the Parseval relation)

$$\langle x, y\rangle = \sum_{k}\langle x, e_k\rangle\overline{\langle y, e_k\rangle}$$

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**Proof.** If the Parseval relation (3) shown as

$$\sum_k |\langle x, e_k\rangle|^2 = \|x\|^2$$

holds for any $x$ in Hilbert space $H$, then for any $x, y$, we have$\|x+y\|^2 = \sum_k |\langle x+y, e_k\rangle|^2$. Note that

$$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2,$$

and

$$\begin{aligned}\sum_k |\langle x+y, e_k\rangle|^2 &= \sum_k (\langle x, e_k\rangle + \langle y, e_k\rangle)\overline{(\langle x, e_k\rangle + \langle y, e_k\rangle)} \\ &= \sum_k |\langle x, e_k\rangle|^2 + \langle x, e_k\rangle\overline{\langle y, e_k\rangle} + \langle y, e_k\rangle\overline{\langle x, e_k\rangle} + |\langle x, e_k\rangle|^2 \\ &= \|x\|^2 + 2Re\langle x, e_k\rangle\overline{\langle y, e_k\rangle} + \|y\|^2.\end{aligned}$$

Then, $Re\langle x, y\rangle = Re\langle x, e_k\rangle\overline{\langle y, e_k\rangle}$. Replacing $y$ by $iy$, we have

$$-Im\langle x, y\rangle = Re\langle x, iy\rangle = Re\langle x, e_k\rangle\overline{\langle iy, e_k\rangle} = -Im\langle x, e_k\rangle\overline{\langle y, e_k\rangle}$$

Therefore, $\langle x, y\rangle = \sum_k \langle x, e_k\rangle\overline{\langle y, e_k\rangle}$. □

**P175, 5** Show that an orthonormal family $(e_\kappa)$, $\kappa \in I$, in a Hilbert space $H$ is total if and only if the relation in Prob. 4 holds for every $x$ and $y$ in $H$.

**Proof.** As shown in Prob. 4,

$$\langle x, y\rangle = \sum_\kappa \langle x, e_\kappa\rangle\overline{\langle y, e_\kappa\rangle} \quad \text{if and only if} \quad \|x\|^2 = \sum_{\kappa\in I} |\langle x, e_\kappa\rangle|^2$$

By Theorem 3.6-3, Hilbert space $H$ is total if and only if the relation $\|x\|^2 = \sum_{\kappa\in I} |\langle x, e_\kappa\rangle|^2$ holds for every $x$ so that the relation in Prob. 4 holds for every $x$ and $y$ in $H$. □