Suggested Solution to Homework 3

Yu Mei[†]

P114, 15.(Half Space) Let $f \neq 0$ be a bounded linear functional on a real normed space X. Then for any scalar c we have a hyperplane $H_c = \{x \in X | f(x) = c\}$, and H_c determines the two half spaces

$$X_{c1} = \{x \in X | f(x) \leq c\} \quad \text{and} \quad X_{c2} = \{x \in X | f(x) \geq c\}$$

Show that the closed unit ball lies in X_{c1} where $c = \|f\|$, but for no $\varepsilon > 0$, the half space X_{c1} with $c = \|f\| - \varepsilon$ contains that ball.

Proof. For any $x \in \tilde{B}(0; 1) := \{x \in X | \|x\| \leq 1\}$,

$$f(x) \leq \|f\|\|x\| = \|f\|.$$

So, $x \in X_{c1}$,i.e. the closed ball lies in X_{c1}.
Since $\|f\| = \sup\limits_{\|x\|=1} |f(x)|$, then for any $\varepsilon > 0$, there exist a x_0 with $\|x_0\| = 1$ such that

$$|f(x)| > \|f\| - \varepsilon.$$

So, for no $\varepsilon > 0$, the half space X_{c1} with $c = \|f\| - \varepsilon$ contains the closed ball. □

P225, 14.(Hyperplane) Show that for any sphere $S(0; r)$ in a normed space X and any point $x_0 \in S(0; r)$ there is a hyperplane $H_0 \ni x_0$ such that the ball $\tilde{B}(0; r)$ lies entirely in one of the two half spaces determined by H_0.

Proof. Since $S(0; r) \ni x_0 \neq 0$, it follows from Theorem 4-3-3 in the textbook that there exists a bounded linear functional $\tilde{f}$ such that $\tilde{f}(x_0) = \|x_0\|$ and $\|\tilde{f}\| = 1$. Set $H_0 = \{x \in X | \tilde{f}(x) = r\}$. It is clear that $x_0 \in H_0$. Moreover, for any $x \in \tilde{B}(0; r)$, $|f(x)| \leq \|f\|\|x\| \leq r$. Therefore, $\tilde{B}(0; r)$ lies entirely in the half plane $X_r = \{x \in X | f(x) \leq r\}$ which determined by H_0. □

P225, 15. If x_0 in a normed space X is such that $|f(x_0)| \leq c$ for all $f \in X'$ of norm 1, show that $\|x_0\| \leq c$.

Proof. If $x_0 = 0$, then it is obvious that $\|x_0\| = 0 \leq c$. For $x_0 \neq 0$, by Theorem 4-3-3 in the textbook, there exist a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = 1$ and $\tilde{f}(x_0) = \|x_0\|$. If $\|x_0\| > c$, then $|\tilde{f}(x_0)| = \|x_0\| > c$, which is a contradiction. Hence $\|x_0\| \leq c$. □

† Email address: ymei@math.cuhk.edu.hk. (Any questions are welcome!)