Suggested Solution to Homework 1

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P70, 3. In ℓ^∞, let Y be the subset of all sequences with only finitely many nonzero terms. Show that Y is a subspace of ℓ^∞ but not a closed subspace.

Proof.

(1) Let $x = \{\xi_j\}, y = \{\eta_j\}$ be any two elements in $Y \subset \ell^\infty$. Then there exist $N \in \mathbb{N}^+$ such that

$$\xi_j = \eta_j = 0, \quad \forall j \geq N,$$

otherwise x, y has infinitely many nonzero terms. Moreover, for any j, $|\xi_j| \leq C_x$ and $|\eta_j| \leq C_y$ for some nonnegetive constants C_x, C_y since $x, y \in \ell^\infty$. Hence, for any $\alpha, \beta \in \mathbb{R}$,

$$\alpha\xi_j + \beta\eta_j = 0, \quad \forall j \geq N; \quad |\alpha\xi_j + \beta\eta_j| \leq |\alpha|C_x + |\beta|C_y, \quad \forall j \in \mathbb{N}^+,$$

which implies that $\alpha x + \beta y \in Y$. So, Y is a subspace of ℓ^∞.

(2) Y is not a closed subspace. For example, let x_n be a sequence such that

$$x_j^{(n)} = \begin{cases} 1/j, & j \leq n, \\ 0, & j > n. \end{cases}$$

i.e. $x_n == \{1, \cdots, \frac{1}{n}, 0, \cdots\}$. It is clear that $x_n \in Y$. Set x be a sequence in ℓ^∞ such that $x_j = \frac{1}{j}$. Thus,

$$\|x_n - x\|_{\ell^\infty} = \frac{1}{n+1} \to 0, \quad as \quad n \to +\infty.$$

But $x \notin Y$ since it has infinitely many nonzero terms. □

P71, 7. Show that convergence of $\|y_1\| + \|y_2\| + \|y_3\| + \cdots$ may not imply convergence of $y_1 + y_2 + y_3 + \cdots$.

Proof. Consider Y in the above problem. Set $y_n = \{\eta_j^{(n)}\} \in Y$ to be a sequence with

$$\eta_n^{(n)} = 1/n^2, \eta_j^{(n)} = 0, \quad for \quad all \quad j \neq n.$$

Then, for any $n \in \mathbb{N}^+$, $\|y_n\| = 1/n^2$ which implies that $\sum_{n=1}^\infty \|y_n\| = \sum_{n=1}^\infty 1/n^2 < +\infty$.
Set $y = \{1, 1/2^2, \cdots, 1/n^2, \cdots\}$. Then,

$$\|\sum_{j=1}^n y_j - y\|_{\ell^\infty} = \frac{1}{(n+1)^2} \to 0, \quad as \quad n \to +\infty.$$

But y has infinitely many nonzero terms, i.e. $\sum_{n=1}^\infty y_n \notin Y$. So, $\sum_{n=1}^\infty y_n$ does not converge in Y. □

P71, 14, Let Y be a closed subspace of a normed space $(X, \|\cdot\|)$. Show that a norm $\|\cdot\|_0$ on X/Y is defined by

$$\|\hat{x}\|_0 = \inf_{x \in \hat{x}} \|x\|$$

where $\hat{x} \in X/Y$, that is, $\hat{x}$ is any coset of Y.

Proof. Recall that $X/Y = \{\hat{x} | \hat{x} = x + Y, x \in X\}$ and algebraic operations in X/Y are defined as:

$$\alpha\hat{x} = \alpha x + Y; \quad \hat{x} + \hat{z} = x + z + Y.$$

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(1) $\|\hat{x}\|_0 = \inf_{x\in\hat{x}} \|x\| = \inf_{y\in Y} \|x+y\| \geq 0$, since $\|\cdot\|$ is a norm on X.

(2) On the one hand, $\hat{0} = Y$ yields that

$$\|\hat{0}\|_0 = \inf_{y\in Y} \|y\| = \|0\| = 0.$$

On the other hand, if $\|\hat{x}\|_0 = 0$, then

$$\inf_{y\in Y} \|x+y\| = \inf_{y\in Y} \|x-y\| = 0$$

which implies $x \in \bar{Y}$. Since Y is a closed subspace, then $\bar{Y} = Y$. Thus, $x \in Y$ so that $\hat{x} = 0$. Hence $\|\hat{x}\|_0 = 0$ if and only if $\hat{x} = \hat{0}$

(3) If $\alpha = 0$, then $\|\alpha\hat{x}\|_0 = \inf_{y\in Y} \|0x+y\| = 0$. For any $\alpha \in \mathbb{R} - \{0\}$, it holds that

$$\|\alpha\hat{x}\|_0 = \inf_{y\in Y} \|\alpha x + y\| = \inf_{y\in Y} \|\alpha(x+\frac{y}{\alpha})\| = |\alpha| \inf_{y\in Y} \|x+\frac{y}{\alpha}\| = |\alpha| \inf_{y\in Y} \|x+y\| = |\alpha|\|\hat{x}\|_0,$$

since Y is a subspace which yields that $\inf_{y\in Y} \|x+\frac{y}{\alpha}\| = \inf_{y\in Y} \|x+y\|$.

(4) For any $\hat{x}, \hat{z} \in X/Y$, by the definition of $infimum$, for any $\varepsilon > 0$, there exist $y_1, y_2 \in Y$ such that

$$\|x+y_1\| \leq \|\hat{x}\|_0 + \varepsilon, \|z+y_2\| \leq \|\hat{z}\|_0 + \varepsilon.$$

Thus,

$$\|x+z+y_1+y_2\| \leq \|x+y_1\| + \|z+y_2\| \leq \|\hat{x}\|_0 + \|\hat{z}\|_0 + 2\varepsilon.$$

which implies that

$$\|\hat{x}+\hat{z}\|_0 = \inf_{y\in Y} \|x+z+y\| \leq \|\hat{x}\|_0 + \|\hat{z}\|_0 + 2\varepsilon.$$

Since ε is arbitrary, it holds that

$$\|\hat{x}+\hat{z}\|_0 \leq \|\hat{x}\|_0 + \|\hat{z}\|_0.$$

□