1 (2021 Home Test 1 Q1)**.** Let $f(x) = \text{sgn}(\sin \frac{\pi}{x})$ for $x \neq 0$ and $f(0) = 0$, where sgn denotes the sign function. Show that f is Riemann integrable over $[-1, 1]$ and find $\int_{-1}^{1} f(x)dx$.

Solution. Note it is not hard to see that f is an odd function over $[-1, 1]$, that is, $f(-x) = -f(x)$ for all $x \in [-1, 1]$. Therefore for all partition $P \subset [0, 1]$, considering $-P \subset [-1, 0]$ to be a partition over the other interval, we have that $U(f, P) = -L(f, -P)$ and $L(f, P) = U(f, -P)$. It is also not hard to see that there is a one-to-one correspondence between partitions over $[0, 1]$ and $[-1, 0]$ respecting refinements of partitions. Therefore, we have $\overline{\int_0^1} f = -\underline{\int_{-1}^{0}} f$ and $\underline{\int_0^1} f = -\overline{\int_{-1}^{0}} f$ by taking limit of nets. Hence it clearly suffices to show that the restriction $f\mid_{[0,1]}$ is Riemann integrable.

To this end, observe that $f(x) = 0$ on $[0, 1]$ if and only if $x = 0$ or $x = 1/n$ for some $n \in \mathbb{N}$. In addition, f is constant on the open intervals $(1/(n+1), 1/n)$ for all $n \in \mathbb{N}$. Let (ϵ_n) be a sequence of real numbers such that $0 < \epsilon_n < 1/2n$ and $\epsilon_n < 1/n^2$ for all $n \in \mathbb{N}$. Then we consider for all $n \in \mathbb{N}$ the partitions

$$P_n := \{0, \frac{1}{n} - \epsilon_n, \frac{1}{n} + \epsilon_n, \cdots, \frac{1}{2} - \epsilon_n \frac{1}{2} + \epsilon_n, 1 - \epsilon_n, 1\} =: \{x_i\}_{i=1}^{k}$$

The choice of (ϵ_n) makes the elements in P_n increase strictly from left to right as written above. It is not hard to see that f is constant on the intervals $[x_{i-1}, x_i]$ except for $[0, \frac{1}{n} - \epsilon_n], [1 - \epsilon_n, 1]$ and $[\frac{1}{p} - \epsilon_n, \frac{1}{p} + \epsilon_n]$ where $1 < p < n$. Hence, we have

$$U(f, P_n) - L(f, P_n) \leq \sum_{i=1}^{n} \text{diam}(f[\frac{1}{i} - \epsilon_n, \frac{1}{i} + \epsilon_n])2\epsilon_i \leq \sum_{i=1}^{n} 4\epsilon_n \leq 4n\epsilon_n \leq 4n/n^2 = 4/n$$

for all $n \in \mathbb{N}$. It is not hard to see that this implies $f \in \mathcal{R}([0, 1])$. By the remark on the first paragraph, it follows that $f \in \mathcal{R}([-1, 0])$ and so $f \in \mathcal{R}([-1, 1])$. In addition. we have $\int_0^1 f = -\int_1^0 f$ from the first paragraph as f is odd. Therefore $\int_{-1}^{1} f = \int_{-1}^{0} f + \int_0^1 f = 0$.

Remark. In fact one can observe that for all $\epsilon > 0$ we have $f \in \mathcal{R}([-1, -\epsilon])$ and $f \in \mathcal{R}([1, \epsilon])$ since functions with finitely many continuity are Riemann integrable. This gives another way to show that f is integrable.

2 (2021 Home Test 1 Q2)**.** Let f be a continuous real-valued function defined on $\mathbb{R}$.

(a) Suppose that there are constants c_0 and c_1 such that

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x}{x} = 0.$$

Show that $f'(0)$ exists.

(b) Suppose that f is a C^1-function and there are constants c_0, c_1 and c_2 such that

$$\lim_{x\to 0} \frac{f(x) - c_0 - c_1 x - c_2 x^2}{x^2} = 0.$$

Does it imply that the second derivative of f at 0 exist? Prove your assertion.

Solution.

a. Write $g(x) := \frac{f(x) - c_0 - c_1 x}{x}$ for $x \neq 0$. Then for $x \neq 0$, we have $xg(x) = f(x) - c_0 - c_1 x$. This implies $f(0) = c_0$ as f is continuous. It is then not hard to see that $g(x) + c_1 = \frac{f(x) - f(0)}{x}$ for $x \neq 0$ and so $f'(0) = c_1$. It particular, $f'(0)$ exists.

b. No. Consider $c_0 = c_1 = c_2 = 0$ and consider $f(x) = x^3 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Then $f'(x) = 3x^2 \sin(1/x) - x\cos(1/x^2)$ for $x \neq 0$ and $f'(0) = 0$. Note that f' is continuous as $\lim_{x\to 0} f'(x) = 0$. Nonetheless, $f'(x)$ does not have derivative at 0.

Remark. It is incorrect to use L'Hospital Rule on the limit of part (b) so that the result on part (a) could be used.

3 (2021 Home Test 1 Q3)**.** Let $f : (0, 1) \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} \frac{1}{p} & \text{if } \ x = \frac{q}{p} \text{ and } p, q \text{ are relatively prime positive integers;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

a) Describe the continuity of f.

b) Describe the differentiability of f.

Justify your answer by using the definitions.

Solution.

a. This was come across in 2058: f is continuous precisely at the irrationals.

b. Since f is only continuous at irrationals, it suffices to consider differentiability at the irrationals. We proceed to claim that f is no-where differentiable. Suppose not. Then f is differentiable at some irrational c. By approaching c with irrationals, then it must be the case that $f'(c) := \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = 0$. It suffices to show that $f'(c)$ cannot be 0. In particular, we show that there exists a sequence of rational numbers (x_n) such that $x_n \to c$ but $\left|\frac{f(x_n)}{x_n - c}\right| \geq 1$. Enumerate the prime numbers (p_n). We want to find that for large enough n, there exists $0 < q_n < p_n$ with $q_n \in \mathbb{N}$ such that $|x_n - c| < 1/p_n$. This would then imply eventually, we have $\left|\frac{f(x_n)}{x_n - c}\right| \geq 1$. To this end, we consider the (unsolved) inequality

$$\begin{aligned} &|x_n - c| < \frac{1}{p_n} \\ \iff &\left|\frac{q_n}{p_n} - c\right| < \frac{1}{p_n} \\ \iff &-1 + cp_n < q_n < 1 + cp_n \end{aligned}$$

for all $n \in \mathbb{N}$. Note that by the unboundedness of (p_n), or the existence of infinitely many primes, it is clear that there exists $N \in \mathbb{N}$ such that $-1 + cp_n > 0$ and $1 + cp_n < p_n$ for all $n \geq N$. Also with $n \geq N$, we have $(-1 + cp_n, 1 + cp_n) \subset (0, p_n) \subset (0, \infty)$ to be of length 2 and so must contain some $q_n \in \mathbb{N}$. Therefore, we have solved the required inequality for large enough N. It follows there exists a sequence of rational numbers (x_n) in $(0, 1)$ such that $\left|\frac{f(x_n)}{x_n - c}\right| \geq 1$ for large enough n. This implies clearly that $f'(c) \neq 0$ which is a contradiction.

Remark. In a sense, the solution to 3(b) is natural because first it is natural to consider rationals with prime denominators to simplify the question; and secondly the (x_n) we consider is basically just the solution to the desired inequality $\left|\frac{f(x_n)}{x_n - c}\right| \geq 1$. It is a standard technique in analysis to identify inequalities that we want and then solve them (in an $\epsilon - \delta$ argument, we identify the $\epsilon-$inequality that we want and try to solve for a δ).

4 (Motivated from 1920 Home Test 1 Q2)**.** Recall that a function $s : [0,1] \to \mathbb{R}$ is a step function over $[0,1]$ if there exists a partition $P := \{x_i\}_{i=0}^k \subset [0,1]$ such that s is constant over (x_{i-1}, x_i).

(a) Let $f \in \mathcal{R}([0,1])$. Show that there exists a sequence of step functions (s_n) over $[0,1]$ such that $s_n \leq s_{n+1}$ pointwise for all $n \in \mathbb{N}$ and $\lim_n \int_0^1 s_n = \int_0^1 f$.

(b) Let $f \in \mathcal{C}([0,1])$, that is f is continuous. Show that there exists a sequence of step functions (s_n) uniformly approximating f, that is, $\lim_n \sup_{x\in[0,1]} |s_n(x) - f(x)| = 0$. Hence, show that the sequence also satisfies $\lim_n \int_0^1 s_n = \int_0^1 f$.

(c) Suppose $f \in \mathcal{R}([0,1])$. Is it always true that f is uniformly approximated by step functions, that is, can the assumption in (b) be relaxed to only integrable functions?

Solution. We begin with a general observation that every lower (and upper) sum corresponds to an intergral of step function. Let $P := \{x_i\}_{i=1}^k$ be a partition over $[0,1]$. Then $L(f,P) = \sum_{i=1}^k m_i(f,P)(x_{i-1}, x_i)$. Now we define a step function s_P by $s_P \equiv m_i(f,P)$ on (x_{i-1}, x_i). Note that there is a unique way of defining s_P on the end-points $P \subset [0,1]$ such that s_P is right-continuous on $[0,1)$ and left-continuous at 1. We define s_P on the end-points according to that. Then it is clear that by splitting domains (or by Lecture Theorems), we have $\int_0^1 s_P = L(f,P)$. Furthermore, it is not hard it see that for a refinement $Q \supset P$, we have $s_Q \geq s_P$ point-wise everywhere. Now we proceed to do the questions:

a. Let (ϵ_n) be a sequence of decreasing positive number such that $\epsilon_n \downarrow 0$. By considering lower integral, there exists a sequence of partitions (P_n) such that $\int_0^1 f - \epsilon_n < L(f, P_n)$. Now define $Q_n := \bigcup_{i=1}^n P_i$. Then it is not hard to see that (Q_n) is increasing with respect to refinements and we have $\int_0^1 f - \epsilon_n < L(f, Q_n)$. Therefore the step functions defined by $s_n := s_{Q_n}$ according to the way stated in the beginning is point-wise increasing. Furthermore we have $\int_0^1 f - \epsilon_n < \int_0^1 s_n = L(f, Q_n)$. This clearly implies that $\lim_n \int_0^1 s_n = \int_0^1 f$ as $n \to \infty$.

b. Let (ϵ_n) be a sequence of decreasing positive number such that $\epsilon_n \downarrow 0$. By compactness, f is uniformly continuous. Hence, there exists $\delta_n > 0$ such that $|f(x) - f(y)| < \epsilon_n$ when $|x - y| < \delta_n$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ choose a partition $P_n := \{x_i^n\}_{i=1}^k \subset [0,1]$ such that $\max_{i=1}^k \left|x_i^n - x_{i-1}^n\right| < \delta_n$. Define s_n to be some step function such that s_n is right continuous on $[0,1)$ and left continuous at 1 such that $s_n \equiv c_{i,n}$ on (x_{i-1}^n, x_i^n) for some $c_{i,n} \in (x_{i-1}^n, x_i^n)$. It follows clearly that $\sup_{x\in[0,1]} |s_n(x) - f(x)| \leq \epsilon_n$ for all $n \in \mathbb{N}$. Hence, (s_n) approximates f uniformly.

   Next we show that $\int_0^1 s_n \to \int_0^1 f$. This follows because we have for all $n \in \mathbb{N}$ that

$$\left|\int_0^1 s_n - \int_0^1 f\right| = \left|\int_0^1 (s_n - f)\right| \leq \int_0^1 |s_n - f| \leq \sup_{x\in[0,1]} |s_n(x) - f(x)|$$

c. No. We claim that that the indicator function $f := \mathbb{1}_{\{\frac{1}{n}:n\in\mathbb{N}\}}$ cannot be uniformly approximated by step functions. Suppose not. Then there exists a step function $s := \sum_{i=1}^k c_i \mathbb{1}_{(x_i, x_{1-i})}$ where $c_i \in \mathbb{R}$ and $\{x_i\} \subset P$ is a partition such that $\sup_{x\in[0,1]} |f(x) - s(x)| < \frac{1}{3}$. It follows that $|f(x) - s(x)| \leq \frac{1}{3}$ for all $x \in (0 =: x_0, x_1)$. In other words, we have $|c_1 - f(x)| \leq \frac{1}{3}$ for all $x \in (0, x_1)$. Nonetheless, note that f attains both 0 and 1 infinitely over $(0, x_1)$. Therefore contradiction arises as it cannot happen at the same time that $|c_1| \leq \frac{1}{3}$ and $|c_1 - 1| \leq \frac{1}{3}$ by the triangle inequality.