Unless otherwise specified, $I \subset \mathbb{R}$ is an open interval.

Definition 1.1. Let $f : I \to \mathbb{R}$ be a function.

- We say that f is differentiable at $c \in I$ if $f'(c) := \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} \in \mathbb{R}$ exists. In this case, we call $f'(c)$ the derivative of f at c
- We say that f is differentiable on I if f is differentiable at all $c \in I$. In that case we call $f' : I \to \mathbb{R}$ the derivative of f over I.

Practice Lv 1

1. Let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that f is continuous at c.

2. Let $f : I \to \mathbb{R}$ be a function. Show that the following are equivalent:

   i. f is differentiable at $c \in I$

   ii. There exists $r > 0$ and a function $\phi : (c - r, c + r) \subset I \to \mathbb{R}$ such that ϕ is continuous at c and
   $$f(x) - f(c) = \phi(x)(x - c)$$
   for all $x \in (c - r, c + r)$. We call such ϕ to be *locally defined* at c.

3. Let $f, g : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that $f + g$ and fg are differentiable at c

   (a) by definition, and

   (b) by Q2

4. Let $f, g : I \to I$ be two functions such that f is differentiable at $c \in I$ and g is differentiable at $f(c) \in I$. Show that $g \circ f$ is differentiable at c.

Practice Lv 2

5. (P.171 Q10) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := \begin{cases} x^2 \sin\big(1/x^2\big) & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that

   (a) g is a differentiable function on $\mathbb{R}$.

   (b) g' is not bounded on $[-1, 1]$

   (You may assume the differentiability of sine functions)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := |\sin(x)|$. Find all points at which f is not differentiable. Explain your answer.

7. Recall that $f : I \to \mathbb{R}$ is said to be Lipschitz function if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a function.

   (a) Suppose f is Lipschitz and differentiable. Show that f' is bounded.

   (b) Can the Lipschitz assumption in part (a) be omitted? Explain your answer and give counter-examples if necessary.

8. Let $f : I \to \mathbb{R}$ where I is bounded. Suppose f is differentiable and f' is uniformly continuous. Show that f is Lipschitz.
   (Hint: Show that f' is bounded first.)

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, that is, for all $x, y \in \mathbb{R}$ and $t \in [0, 1]$, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

   (a) Let $x, y, z \in \mathbb{R}$ be such that $x < y < z$. Show that we have

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(z)}{x - z}$$

   (b) Show that for all $c \in \mathbb{R}$ the right limit $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ exists; in particular it does **<u>not</u>** diverge to infinities.

   (c) Show that $\lim_{x \to c} f(x) = f(c)$ for all $c \in \mathbb{R}$.

   (Hint: It is better for you to first think about the meaning (e.g. graphically) of a convex function.)