1. Let f be a real-valued function defined on $[0, \infty)$. Prove or disprove the following statements:

i. If $\int_0^\infty f(x)dx$ and $\lim_{x\to\infty} f(x)$ both exists, then $\lim_{x\to\infty} f(x) = 0$

ii. If $\int_0^\infty f(x)dx$ exists, then $\lim_{x\to\infty} f(x)$ exists.

Solution.

i. The statement is true. Suppose not. Then $\lim_{x\to\infty} f(x) \neq 0$. WLOG suppose $\lim_{x\to\infty} f(x) = \epsilon > 0$ (since $\lim_{x\to\infty} f(x)$ exists). Then $f(x) > \epsilon/2$ for $x \geq M$. Hence it follows that for all $n \in \mathbb{N}$, we have

$$\int_M^{M+n} f(x)dx \geq \int_M^{M+n} \frac{\epsilon}{2}dx = n\frac{\epsilon}{2}$$

It follows we have for all $n \in \mathbb{N}$ that

$$\int_0^{M+n} f(x)dx = \int_0^M f(x)dx + \int_M^{M+n} f(x)dx \geq \int_0^M f(x)dx + n\frac{\epsilon}{2}$$

This implies that $\lim_n \int_0^{M+n} f(x)dx = \infty$ as $\lim n\epsilon/2 = \infty$. It follows from the sequential criteria that $\int_0^\infty f(x)dx = \infty$, which is a contradiction.

ii. The statement is false. Consider $f(x) = 1/(x+1)^2$ for $x \geq 0$. Note that f is continuous on $[0, b]$ for all $b \in (0, \infty)$. It follows from FTC that $\int_0^b f(x)dx = -1/(x+1)]_0^b = 1 - 1/b$. By taking limits, we have $\int_0^\infty f(x)dx = \lim_{b\to\infty} \int_0^b f(x)dx = 1$.
Now we consider another function g such that $g(x) = f(x)$ for all $x \geq 0$ and $x \notin \mathbb{N}$ and $g(n) = 1$ for $n \in \mathbb{N}$. Note that for all $b > 0$, $g = f$ except for finitely many points on $[0, b]$. Hence, we have $\int_0^b g = \int_0^b f$. It follows by taking limits that $\int_0^\infty g = \lim_{b\to\infty} \int_0^b f = \int_0^\infty f = 1$, that is, $\int_0^\infty g(x)dx$ exists. However $\lim_{x\to\infty} f(x)$ does not exists since $g(x_n) \to 1$ where $x_n := n$ but $g(x_n + 1/2) = f(x_n + 1/2) = 1/(n + 1/2)^2 \to 0$.

Remark.

- All of you correctly identified the truth of the statements. Well done!
- Many of you considered f in part (ii) to be the zero function, which is clearly Okay and simpler than the above solution. Meanwhile, a number of you considered functions with "hats" of diminishing width, which is also OK.
- Some of you confused the definition of improper integrals with the ordinary integral: ordinary integrals could be verified by an ϵ argument concerning upper and lower sum of partitions. However the case for improper integrals is not (as we may not be able to define upper/lower sums if functions or domains are unbounded). The latter is *defined* as the limit of ordinary integrals on compact intervals. In particular, theorems that are true for the ordinary integral may not be valid for improper integrals; one has to give some verification before using similar theorems, like the Lebesgue criteria.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Write $M := \sup_{x\in\mathbb{R}} |f(x)|$. For all $\lambda > 0$, we define

$$\psi_\lambda(x) := \inf\{g(x) : g \textit{ is a } \lambda\textit{-Lipschitz function on } \mathbb{R} \textit{ and } g \geq f \textit{ on } \mathbb{R}\}$$

for all $x \in \mathbb{R}$. Write $\psi_0(x) := M$ for all $x \in \mathbb{R}$.
Suppose for all $t > 0$, there exists $\lambda > 0$ such that $\psi_\lambda(x) - f(x) < t$ for all $x \in \mathbb{R}$. With the assumption, we define for all $t > 0$ that

$$\tau(t) := \inf\{\lambda > 0 : \psi_\lambda(x) - f(x) < t \textit{ for all } x \in \mathbb{R}\}$$

We also define

$$\phi(x) := \int_0^1 \psi_{\tau(t)}(x)dt \tag{1}$$

for all $x \in \mathbb{R}$

i. Show that for all $\lambda > 0$ that ψ_λ is a $\lambda-$Lipschitz function on $\mathbb{R}$.

ii. Show that the impropert integral in Eq(1) exists for all $x \in \mathbb{R}$, that is, the function $t \in [c, 1] \mapsto \psi_{\tau(t)}(x)$ is Riemann integrable for all $c \in (0, 1]$ and $\lim_{c\to 0^+} \int_c^1 \psi_{\tau(t)}(x)dt$ exists.

iii. Show that the function ϕ is bounded and uniformly continuous on $\mathbb{R}$.

Solution.

i. Fix $\lambda > 0$. Fix we show that $\psi_\lambda(x)$ is well-defined for all $x \in \mathbb{R}$. Set $g(x) := M$ for all $x \in \mathbb{R}$. Then it is clear that g is $\lambda-$ Lipschitz for all $\lambda > 0$ since it is a constant function. It is also clear that $g \geq f$ on $\mathbb{R}$. Therefore, $\psi_\lambda(x)$ is finite for all $x \in \mathbb{R}$.
Next, we show that ψ_λ is $\lambda-$Lipschitz. Let $x, y \in \mathbb{R}$. Let $\epsilon > 0$. Then there exists g, h λ- Lipschitz functions such that $\psi_\lambda(x) + \epsilon > g(x)$ and $\psi_\lambda(y) + \epsilon > h(y)$ by the definition of ψ_λ. Hence, we have

$$\psi_\lambda(y) - \psi_\lambda(x) - \epsilon \leq \psi_\lambda(y) - g(x) \leq g(x) - g(y) \leq \lambda|x - y|$$
$$\psi_\lambda(x) - \psi_\lambda(y) - \epsilon \leq \psi_\lambda(x) - h(y) \leq h(x) - h(y) \leq \lambda|x - y|$$

Combining the two, we have $|\psi_\lambda(x) - \psi_\lambda(y)| - \epsilon \leq \lambda|x - y|$. The result follows as $\epsilon \to 0$.

ii. Fix $x \in \mathbb{R}$. Write $\alpha_x(t) := \psi_{\tau(t)}(x)$ for $t > 0$.
Claim. We have $\alpha_x(t)$ is increasing for $t > 0$
Proof of claim. To begin with, we show that τ is decreasing on $(0, \infty)$. Suppose $t_1 \leq t_2 \in (0, 1)$. Then we have

$$\{\lambda > 0 : \psi_\lambda(x) - f(x) < t_1 \textit{ for all } x \in \mathbb{R}\} \subset \{\lambda > 0 : \psi_\lambda(x) - f(x) < t_2 \textit{ for all } x \in \mathbb{R}\}$$

It follows clearly that $\tau(t_2) \leq \tau(t_1)$ by taking infimums. Next, we show that $\psi_\lambda(x)$ is decreasing for $\lambda \geq 0$. This is because if $0 < \lambda_1 \leq \lambda_2$ then we have

$$\{g(x) : g \textit{ is a } \lambda_1\textit{-Lipschitz function on } \mathbb{R} \textit{ and } g \geq f \textit{ on } \mathbb{R}\}$$
$$\subset\{g(x) : g \textit{ is a } \lambda_2\textit{-Lipschitz function on } \mathbb{R} \textit{ and } g \geq f \textit{ on } \mathbb{R}\}$$

By taking infimums, we have $\psi_{\lambda_2}(x) \leq \psi_{\lambda_1}(x)$. The case where $\lambda_1 = 0$ is obvious using the definition of ψ_0. Combining the two, we have that $\alpha_x(t) := \psi_{\tau(t)}(x)$ is increasing for $t > 0$. □

By the claim, α_x is increasing on $(0, 1]$. In particular, it is increasing on $[c, 1]$ for all $c \in (0, 1)$. Hence $\alpha_x \in \mathcal{R}([c, 1])$ as monotone functions over compact intervals are Riemann integrable.
Next, we give a bound for α:
Claim. We have $|\alpha_x(t)| \leq M$ for all $t > 0$.
Proof of claim. Fix $\lambda \geq 0$. Note that $\psi_\lambda(x) \geq f(x) \geq -M$ by the definition of ψ_λ. In addition, we have shown in the proof of the preivous claim that $\psi_\lambda(x) \leq \psi_0(x) = M$ (since the constant function $g \equiv M$ on $\mathbb{R}$ is λ-Lipschitz for all $\lambda > 0$ with $g \geq f$ on $\mathbb{R}$). It follows that $|\psi_\lambda(x)| \leq M$ for all $\lambda \geq 0$. Hence, we clearly have $|\alpha_x(t)| \leq M$ on $t > 0$ □

To show that $\lim_{c\to 0^+} \int_c^1 \alpha_x(t)dt$ exists. We consider $\overline{\alpha}_x(t) := \alpha_x(t) + M \geq 0$ for all $t > 0$. Note that $\overline{\alpha}_x(t)$ is non-negative and increasing on $(0, 1]$. Define $F_x(c) := \int_c^1 \overline{\alpha}_x(t)dt$ for all $c > 0$. It follows that F_x is decreasing for $c \in (0, 1]$ by splitting the domain of integration. Furthermore, $0 \leq F_x(c) \leq \int_c^1 |\overline{\alpha}_x(t)|dt \leq 2M$ for all $c \in (0, 1]$. It follows from the bounded monotone theorem that $\lim_{c\to 0^+} F_x(c)$ exists. In particular, by linearity of integrals and limits, $\lim_{c\to 0^+} \int_c^1 \alpha_x(t)dt$ exists.

iii. First we show that ϕ is bounded. Fix $x \in \mathbb{R}$. Using the notations in part (ii), we have $\phi(x) = \lim_{c \to 0^+} F_x(c)$. As $0 \le F_x(c) \le 2M$ for all $c \in (0, 1]$, we have $0 \le \phi(x) \le 2M$. The result follows as x is arbitrary.
Next we show the uniform continuity. Let $\epsilon > 0$. Let $x, y \in \mathbb{R}$. Pick $c \in (0, \epsilon)$ such that

$$|\phi(x) - F_c(x)|, |\phi(y) - F_c(y)| \le \epsilon$$

Then, note that

$$\begin{aligned}
|F_c(x) - F_c(y)| &= \left| \int_c^1 \psi_{\tau(t)}(x) - \psi_{\tau(t)}(y) dt \right| \le \int_c^1 \left| \psi_{\tau(t)}(x) - \psi_{\tau(t)}(y) \right| dt \\
&= \int_c^\epsilon \left| \psi_{\tau(t)}(x) - \psi_{\tau(t)}(y) \right| dt + \int_\epsilon^1 \left| \psi_{\tau(t)}(x) - \psi_{\tau(t)}(y) \right| dt \\
&\le \int_c^\epsilon |\alpha_x|(t) + |\alpha_y|(t) dt + \int_\epsilon^1 \tau(t)|x - y| dt \\
&\le 2M\epsilon + \int_\epsilon^1 \tau(\epsilon) dt |x - y| \\
&\le 2M\epsilon + \tau(\epsilon)|x - y|
\end{aligned}$$

Choose $\delta := \min\{1, \epsilon/\tau(\epsilon)\}$ (we set $\epsilon/0 := \infty$ if necessary). Then it follows that $|F_c(x) - F_c(y)| \le (2M+1)\epsilon$ as $|x - y| \le \delta$. By triangle inequalities, we further have $|\phi(x) - \phi(y)| \le (2M + 3)\epsilon$ when $|x - y| \le \epsilon$. It follows that ϕ is uniform continuous on $\mathbb{R}$.