In the following, unless otherwise specificed, if $f : A \to \mathbb{R}$ is a bounded function on $A \subset \mathbb{R}$, then we denote $\|f\|_\infty := \sup\{|f(x)| : x \in A\}$.

1. Define $f_n(x) := \frac{x^n}{(1+x^n)}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Find the pointwise limit of (f_n) for $x \geq 0$.

Solution. Fix $x \geq 0$. Suppose $x = 0$. Then $f_n(0) = 0$ for all $n \in \mathbb{N}$. Therefore $f_n(0) \to 0$. Suppose $1 > x > 0$. Then $\lim_n x^n = 0$. It follows that $\lim_n f_n(x) = 0$. Suppose $x > 1$. Then $0 < x^{-1} < 1$. Note that $f_n(x) = \frac{x^n}{1+x^n} = \frac{1}{x^{-n}+1}$ for all $n \in \mathbb{N}$. It follows that $\lim_n f_n(x) = \frac{1}{0+1} = 1$. Suppose $x = 1$. Then $f_n(x) = 1/2$ for all $n \in \mathbb{N}$. It follows that $\lim_n f_n(1) = 1/2$. Hence, define $f(x) := \begin{cases} 0 & x \in [0,1) \\ 1/2 & x = 1 \\ 1 & x > 1 \end{cases}$. It follows that $f_n(x) \to f(x)$ pointwise on $x \geq 0$.

2. Consider (f_n) to be the sequence of functions defined in Question 1. Let $b \in (0,1)$.

i. Show that f_n converges uniformly on $[0,b]$

ii. Show that f_n does not converge uniformly on $[0,1]$

Solution.

i. We claim that $f_n \to 0$ on $[0,b]$ uniformly for all $b \in (0,1)$. Fix $b \in (0,1)$. Note that we have

$$0 \leq f_n(x) = \frac{x^n}{1+x^n} \leq \frac{b^n}{1} = b^n$$

for all $x \in [0,b)$ and $n \in \mathbb{N}$. It follows that we have $\sup_{x\in[0,b]} |f_n(x) - 0| \leq b^n$. Note that $\lim b^n = 0$ as $b \in (0,1)$. By squeeze theorem, we have $\lim_n \sup_{x\in[0,b]} |f_n(x) - 0| = 0$. This shows that $f_n \to 0$ uniformly on $[0,b]$.

ii. Define $f : [0,1] \to \mathbb{R}$ by $f(x) := 0$ for all $x \neq 1$ and $f(1) := 1/2$. By Question 1, $f_n \to f$ pointwise on $[0,1]$. It suffices to show that f_n does not converge to f uniformly on $[0,1]$. Note that for all $n \in \mathbb{N}$ take $x_n := 1 - 1/n \in [0,1]$. Then we have

$$f_n(x_n) = \frac{x_n^n}{1+x_n^n} \geq \frac{x_n^n}{1+1} = \frac{1}{2}(1-\frac{1}{n})^n$$

Therefore, we have $\underline{\lim} f_n(x_n) \geq e^{-1}/2$ (note that $e^{-1} = \lim_n(1-1/n)^n$). Note that we have that $\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq |f_n(x_n) - f(x_n)| = f_n(x_n)$. It follows that we have the approximation that $\underline{\lim} \sup_{x\in[0,1]} |f_n(x) - f(x)| \geq e^{-1}/2$. This implies that $\underline{\lim} \sup_{x\in[0,1]} |f_n(x) - f(x)| > 0$ and so f_n does not converge to f uniformly on $[0,1]$.

Remark. In addition to picking $x_n := 1 - 1/n$, one can pick $x_n := (1/2)^{1/n}$. It is also valid to pick (x_n) by considering $\lim_{x\to1^-} f_n(x) = \frac{1}{2}$; we can take x_n such that $f_n(x_n) > 1/4$ from the limit. One can even pick x_n with $f_n(x_n) = 1/4$ by the intermediate value theorem using continuity of f_n.

3. Define $f_n(x) := x + \frac{1}{n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$; define also $f(x) := x$ for all $x \in \mathbb{R}$.

i. Show that $f_n \to f$ uniformly on $\mathbb{R}$.

ii. Show that the sequence (f_n^2) does not converge uniformly on $\mathbb{R}$.

Solution.

i. Note that $f_n(x) - f(x) = \frac{1}{n}$ for all $x \in \mathbb{R}$. It follows that $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \frac{1}{n} \to 0$ as $n \to \infty$. Hence, $f_n \to f$ uniformly on $\mathbb{R}$.

ii. Note that $f_n(x) = (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Note that $f_n^2(x) \to f^2(x)$ pointwise as $f_n(x) \to f(x)$ pointwise. It sufficies to show that (f_n^2) does not converge to f^2 uniformly on $\mathbb{R}$. Let $n \in \mathbb{N}$. Define $x_n := n$. It follows that $f_n^2(x_n) - f^2(x_n) = \frac{2x_n}{n} + \frac{1}{n^2} = 2 + \frac{1}{n^2}$. It follows that

$$\sup_{x\in\mathbb{R}} \left|f_n^2(x) - f^2(x)\right| \geq \left|f_n^2(x_n) - f^2(x_n)\right| = \left|2 + \frac{1}{n^2}\right| \geq 2$$

It follows that $\underline{\lim} \sup_{x\in\mathbb{R}} \left|f_n^2(x) - f^2(x)\right| \geq 2 > 0$ and so f_n^2 does not converge to f^2 uniformly.

4. Let (f_n) and (g_n) be sequences of bounded functions on a subset $A \subset \mathbb{R}$. Suppose $f_n \to f$ and $g_n \to g$ uniformly on A where $f, g : A \to \mathbb{R}$ are functions. Show that $(f_n g_n)$ converges uniformly to fg on A.

Solution. First we proceed with the following claim

Claim. Suppose (f_n) is a sequence of bounded functions such that $f_n \to f$ uniformly on A. Then $\sup_n \|f_n\|_\infty < \infty$ and $\|f\|_\infty < \infty$.

Proof of claim. First we show that $\|f\|_\infty < \infty$. Fix $t \in A$. Note that there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| \leq 1$$

for all $x \in A$ and $n \geq N$. Hence, we have $|f(t)| \leq 1 + |f_N(t)| \leq 1 + \|f_N\|_\infty$ for all $t \in A$. This implies that $\|f\|_\infty \leq 1 + \|f_N\|_\infty$ by taking supremums.
Next, we show that $\sup_n \|f_n\|_\infty < \infty$. Note that from the above we have for all $n \geq N$ and $t \in A$ that $|f_n(t)| \leq 1 + |f(t)| \leq 1 + \|f\|_\infty$. It follows that $\|f_n\|_\infty \leq 1 + \|f\|_\infty$ for all $n \geq N$.
Finally, take $M := \max\{\|f_1\|_\infty, \cdots, \|f_N\|_\infty, 1 + \|f\|_\infty\}$. It clearly follows that $\|f_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Hence, we have $\sup_n \|f_n\|_\infty < \infty$

Now we proceed to the statement in question with the help of the claim. Let $x \in A$. It follows that we have

$$\begin{aligned} |f_n g_n(x) - fg(x)| &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ &\leq \|f_n\|_\infty \|g_n - g\|_\infty + \|g\|_\infty \|f_n - f\|_\infty \\ &\leq \sup_n \|f_n\| \|g_n - g\|_\infty + \|g\|_\infty \|f_n - f\|_\infty \end{aligned}$$

in which the supremums in the third line is well-defined because of the claim. Hence, we have by taking supremum for $x \in A$ that

$$\|f_n g_n - fg\|_\infty \leq \sup_n \|f_n\| \|g_n - g\|_\infty + \|g\|_\infty \|f_n - f\|_\infty$$

Note that $\|g_n - g\|_\infty \to 0$ and $\|f_n - f\|_\infty \to 0$ as $g_n \to g$ and $f_n \to f$ uniformly. It follows from the squeeze theorem that $\|f_n g_n - fg\| \to 0$ as $n \to \infty$.

Common Mistake. Proving $\sup_n \|f_n\|_\infty, \|f\|_\infty, \|g\|_\infty < \infty$ is crucial for this question. In general, a sequence of bounded functions may not be uniformly bounded; the point-wise limit may not be bounded if the convergence is not uniform.