1 (P. 215 Q13)**.** Give an example of a function $f : [a, b] \to \mathbb{R}$ which satisfies the property that $f \in \mathcal{R}([c, b])$ for all $c \in (a, b)$ but $f \notin \mathcal{R}([a, b])$.

Solution. Consider $f : [0, 1] \to \mathbb{R}$ be defined by $f(0) := 0$ and $f(t) = 1/t$ for $t \in (0, 1]$. It is clear that f is continuous on $[t, 1]$ for all $t > 0$. Therefore, $f \in \mathcal{R}([t, 1])$ for all $t > 0$. It is clear that $f \notin \mathcal{R}([0, 1])$ as f is no even bounded on $[0, 1]$.

2 (P. 224 Q14)**.** Show that there does not exist $f \in \mathcal{C}^1([0, 2])$ such that $f(0) = -1, f(2) = 4$ and $f'(x) \leq 2$ for all $x \in [0, 2]$.

Solution. Suppose not. Let $f \in \mathcal{C}^1([0, 2])$ be such that $f(0) = 1, f(2) = 4$ and $f' \leq 2$ on $[0, 2]$. Note $f' \in \mathcal{R}([0, 2])$ and f is an anti-derivative of f'. By FTC we have $f(2) - f(0) = \int_0^2 f' \leq 2(2 - 0) = 4$. However, $f(2) - f(0) = 5 > 4$, which is a contradiction.

Comment. This question can also be done by MVT; hence, we do not need f to be continuously differentiable.

3 (P. 224 Q17)**.** Let $J := [\alpha, \beta]$ and $\phi : J \to \mathbb{R}$ be continuously differentiable on J. Let $f : I \to \mathbb{R}$ be a continuous function on some (compact) interval I with $I \supset \phi(J)$. Define $F(u) := \int_{\phi(\alpha)}^u f(x)dx$ for all $u \in I$ and $H(t) := F(\phi(t))$ for all $t \in J$.

i. Show that $H'(t) = f(\phi(t))\phi'(t)$ for all $t \in J$

ii. Show that
$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = F(\phi(\beta)) = H(\beta) = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$$

Solution.

i. Note that F is differentiable as f is continuous by FTC. In addition ϕ is differentiable. It follows from the chain rule that H is differentiable and we have $H'(t) = F'(\phi(t))\phi'(t)$ for all $t \in J$. Note that $F' = f$ by FTC. It follows that $H'(t) = f'(\phi(t))\phi'(t)$ for all $t \in J$.

ii. The first two equalities are from definitions. It remains to show the last inequality. Note that since f, ϕ' are continuous, it follows that H' is continuous. Therefore, by FTC, we have
$$H(\beta) - H(\alpha) = \int_\alpha^\beta H'(t)dt = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$$
The result follows as $H(\alpha) = F(\phi(\alpha)) = \int_{\phi(\alpha)}^{\phi(\alpha)} f(x)dx = 0$.

Common Mistake. To differentiate this course from Calculus in high school, you are required to state clearly the reasons why the FTC could be applied. Luckily, more than half of you have stated suitable reasons. Keep it up!