1 (P. 215 Q2)**.** Let $h : [0, 1] \to \mathbb{R}$ be defined by

$$h(x) := \begin{cases} x+1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

for all $x \in [0, 1]$. Show that h is **not** Riemann integrable.

Solution. Consider a compact interval $I := [c, d] \subset [0, 1]$ with $c < d$ and $\operatorname{diam} h([c, d]) := \sup_{x,y\in[c,d]} |h(x) - h(y)|$. Then it is clear that $\operatorname{diam} h([c, d]) \leq 1 + d - 0 = 1 + d$ since we have $0 \leq h(x) \leq 1 + d$ for all $x \in [c, d]$. Now consider a sequence (q_n) in $[c, d]$ such that $q_n \to d$ (which exists by density of $\mathbb{Q}$) and any irrational $\alpha \in [c, d]$, then we have $\operatorname{diam} h([c, d]) \geq h(q_n) - h(\alpha) = 1 + q_n$ for all $n \in \mathbb{N}$ and so $\operatorname{diam} h([c, d]) \geq 1 + d$ as $n \to \infty$. It follows that $\operatorname{diam} h([c, d]) = 1 + d$ for all compact interval $I := [c, d]$ with $c < d$.
Now let $P := \{x_i\}_{i=1}^k$ be a partition of $[0, 1]$ then it follows from the above that

$$U(h, P) - L(h, P) = \sum_{i=1}^{k} \omega_i(h, P)\Delta x_i = \sum_{i=1}^{k} \operatorname{diam} h([x_{i-1}, x_i])\Delta x_i = \sum_{i=1}^{k} (1 + x_i)\Delta x_i \geq \sum_{i=1}^{k} 1\Delta x_i = 1 - 0 = 1$$

It follows clearly that h is not Riemann integrable by definition as $U(h, P) - L(h, P)$ cannot be arbitrarily small.

Common Mistake. It is not the case that $\sup h([x, y]) = 1 + c$ for some $c \in \mathbb{Q} \cap (x, y)$ if $y \notin \mathbb{Q}$. Please refer to the above answer regarding how to compute $\sup h([x, y])$ (from the computation of $\operatorname{diam} h([x, y])$. In fact we just need $\sup h([x, y]) \geq 1$ for any interval $[x, y] \subset [0, 1]$, which is a lot easier to show.

2 (P. 215 Q8)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous with $f \geq 0$ pointwise. Suppose $\int_a^b f = 0$. Show that $f \equiv 0$ on $[a, b]$.

Solution. Suppose not. Then $f(c) > 0$ for some $c \in [a, b]$. By continuity, there exists $\frac{b-a}{2} > r > 0$ such that $f > f(c)/2$ on $B_r(c) \cap [a, b]$. Note that $I := B_r(c) \cap [a, b]$ is an interval of length at least r regardless of where c is. Without loss of generality, write $I := (t, t + r) \subset [a, b]$ for $t \in [a, b]$ Then it follows that we have

$$\int_a^b f \overset{(*)}{\geq} \int_t^{t+r} f \geq \int_t^{t+r} \frac{f(c)}{2} \geq \frac{rf(c)}{2} > 0$$

in which $(*)$ follows from splitting $[a, b]$ into intervals together with the non-negativity of f. Hence contradiction arises as $\int_a^b f = 0$.

Common Mistake. It is not sufficient to just consider $f(c) > 0$ and then apply continuity of f to obtain $f > 0$ on $B_r(c)$. Either one has to consider an even smaller compact interval and apply the extreme value theorem, or one bounds instead $f(c) > \epsilon > 0$ for some $\epsilon > 0$ first. I chose $\epsilon := f(c)/2$ in the above solution. The latter "inserting values" technique is very common in analysis.

3 (P. 215 Q9)**.** Show that the continuity assumption in Q2 (textbook Q8) cannot be dropped, that is, find $f \in \mathcal{R}[a, b]\backslash C([a, b])$ such that for some $a < b \in \mathbb{R}$ such that $\int_a^b f = 0$ but $f(x) \neq 0$ for some $x \in [a, b]$.

Solution. Let $c \in [0, 1]$. Consider $f := \mathbb{1}_{\{c\}}$ on $[0, 1]$ the indicator function of c, that is, $f(x) = 1$ if $x = c$ and 0 otherwise for all $x \in [0, 1]$. Then f is clearly not continuous at c and so not continuous on $[0, 1]$. In addition $f \geq 0$ on $[0, 1]$ and $f(c) \geq 0$. However f is constantly 0 except for finitely many (one) point(s) while the constant zero function is clearly Riemann integrable with integral 0. Therefore $f \in \mathcal{R}([0, 1])$ with $\int_0^1 f = 0$.

Remark. Most of you used similar examples (with some stating the Thomae's functions). In fact, any function that is equal to a continuous function except for finitely many points will do. One could refer to HW 4 solution for an $\epsilon-$ argument in showing the Riemann integrability of the counter-example here as well as in computing its integral.