1 (P.207 Q6).

i. Let $f(x) := \begin{cases} 2 & x \in [0,1) \\ 1 & x \in [1,2] \end{cases}$ for all $x \in [0,2]$. Show that $f \in \mathcal{R}[0,2]$ and find $\int_0^2 f$.

ii. Let $h(x) := \begin{cases} 2 & x \in [0,1) \\ 3 & x = 1 \\ 1 & x \in (1,2] \end{cases}$. Show that $h \in \mathcal{R}([0,2])$ and find $\int_0^2 h$.

Solution.

i. Let $1 > \epsilon > 0$. Then consider the partition $P := \{0, 1-\frac{\epsilon}{4}, 1+\frac{\epsilon}{4}, 2\} = \{x_i\}_{i=0}^k$. It follows that

$$\sum_{i=1}^k \omega_i(f,P)(x_i - x_{i-1}) = \operatorname{diam} f([1-\frac{\epsilon}{4}, 1+\frac{\epsilon}{4}])(1+\frac{\epsilon}{4} - (1-\frac{\epsilon}{4})) = (2-1)\frac{\epsilon}{2} < \epsilon$$

By definition of integrability, it follows that $f \in \mathcal{R}([0,2])$. To compute the integral, now consider the partitions $Q_\epsilon := \{0, 1-\frac{\epsilon}{4}, 1+\frac{\epsilon}{4}, 2\} \subset [0.2]$ for all $\epsilon \in (0,1)$. It is easy to see that

$$U(f, Q_\epsilon) = 2 \cdot (1 - \frac{\epsilon}{4}) + 2 \cdot \frac{\epsilon}{2} + 1 \cdot (1 - \frac{\epsilon}{4}) = 3 + \frac{\epsilon}{4}$$
$$L(f, Q_\epsilon) = 2 \cdot (1 - \frac{\epsilon}{4}) + 1 \cdot \frac{\epsilon}{2} + 1 \cdot (1 - \frac{\epsilon}{4}) = 3 - \frac{\epsilon}{4}$$

Note that by definition of integrals, we have that $L(f, Q_\epsilon) \le \int_0^2 f \le U(f, Q_\epsilon)$ for all $\epsilon \in (0,1)$. As $\epsilon \to 0$, we have by Squeeze theorem that $\int_0^2 f = 3$.

ii. Note that $h \in \mathcal{R}([0,2])$ if and only if $f := h\,|_{[0,1]} \in \mathcal{R}([0,1])$ and $g := h\,|_{[1,2]} \in \mathcal{R}([1,2])$. Note that f is constantly 2 on $[0,1]$ except for finitely many (one) point while constant functions are clearly Riemann integrable and have easily computable integrals. It follows that $f \in \mathcal{R}([0,1])$ and we have $\int_0^1 f = \int_0^1 2 = 2$ (cf. Theorem 7.1.3). By similar argument, we can conclude that $\int_1^2 g = \int_1^2 1 = 1$. Therefore $h \in \mathcal{R}([0,2])$ by the initial remark with the integral being $\int_0^2 h = \int_0^1 f + \int_1^2 g = 2 + 1 = 3$.

Remark. The two proof methods in (i) and (ii) can be used to prove both questions.

2 (P. 207 Q8). Let $a < b$. Let $f \in \mathcal{R}([a,b])$. Suppse $|f| \le M$ point-wise on $[a,b]$ for some $M > 0$. Show that

$$\left| \int_a^b f \right| \le M(b-a)$$

Solution. Let $g := M \cdot \mathbb{1}_{[a,b]} : [a,b] \to \mathbb{R}$, that is, g is constantly M on $[a,b]$. It follows from the assumption that $|f| \le g$ on $[a,b]$ pointwise. Note that $g \in \mathcal{R}([a,b])$ clearly with $\int_a^b g = \int_a^b M = M(b-a)$. It follows from the triangle inequality and monotonicity of integrals that we have

$$\left| \int_a^b f \right| \le \int_a^b |f| \le \int_a^b g = M(b-a)$$

Alternatively, one can proceed by considering the definitions of upper and lower sums. Note that we have $-M \le f \le M$ point-wise by the assumption. Let $P := \{x_i\}_{i=1}^k \subset [a,b]$ be a partition. Then we have

$$U(f,P) := \sum_{i=1}^k \sup f([x_{i-1}, x_i])(x_i - x_{i-1}) \le \sum_{i=1}^k M(x_i - x_{i-1}) = M(b-a)$$
$$L(f,P) := \sum_{i=1}^k \inf f([x_{i-1}, x_i])(x_i - x_{i-1}) \ge \sum_{i=1}^k -M(x_i - x_{i-1}) = -M(b-a)$$

Since P is arbitrary, by consider net convergence (or simply supremums/ infimums, we have

$$-M(b-a) \le \lim_P L(f,P) = \underline{\int_a^b} f = \int_a^b f = \overline{\int_a^b} f = \lim_P U(f,P) \le M(b-a)$$

The result follows clearly.

3 (P. 207 Q13)**.** Let $a < b \in \mathbb{R}$. Fix $c < d \in [a, b]$. Define $\phi(x) := \begin{cases} \alpha & x \in [c, d] \\ 0 & x \notin [c, d] \end{cases}$ for all $x \in [a, b]$ for some real number $\alpha > 0$.

i. Show that $\phi \in \mathcal{R}([a, b])$

ii. Show that $\int_a^b \phi = \alpha(d - c)$

Solution. The proof here is similar to Q1. We demonstrate an $\epsilon-$ argument here.

i. We shall only show the case for $c, d \in (a, b)$. The case that at least one of c, d is an endpoint is similar. Let $\epsilon > 0$ such that $\epsilon < c - a, b - d, \frac{d-c}{2}$. Consider the partition $P_\epsilon := \{a, c - \frac{\epsilon}{2}, c + \frac{\epsilon}{2}, d - \frac{\epsilon}{2}, d + \frac{\epsilon}{2}, b\} =: \{x_i^\epsilon\}_{i=1}^k$. The bound of ϵ ensures that the listed elements of P_ϵ strictly increase from left to right. It follows clearly that we have

$$\begin{aligned} U(\phi, P_\epsilon) &= \alpha \cdot (c + \frac{\epsilon}{2} - (c - \frac{\epsilon}{2})) + \alpha \cdot (d - \frac{\epsilon}{2} - (c + \frac{\epsilon}{2})) + \alpha \cdot (d + \frac{\epsilon}{2} - (d - \frac{\epsilon}{2})) \\ &= \alpha \cdot \epsilon + \alpha(d - c - \epsilon) + \alpha \cdot \epsilon = \alpha(d - c + \epsilon) \\ L(\phi, P_\epsilon) &= 0 \cdot (c + \frac{\epsilon}{2} - (c - \frac{\epsilon}{2})) + \alpha \cdot (d - \frac{\epsilon}{2} - (c + \frac{\epsilon}{2})) + 0 \cdot (d + \frac{\epsilon}{2} - (d - \frac{\epsilon}{2})) \\ &= \alpha(d - c - \epsilon) \end{aligned}$$

Hence, we have by definition that

$$L(\phi, P_\epsilon) \le \underline{\int_a^b} \phi \le \overline{\int_a^b} \phi \le U(\phi, P_\epsilon)$$

for all $\epsilon > 0$ and $\epsilon < c - a, b - d, \frac{d-c}{2}$. As $\epsilon \to 0$, we clear have $\alpha(d - c) \le \underline{\int_a^b} \phi \le \overline{\int_a^b} \phi \le \alpha(d - c)$. This shows that $\overline{\int_a^b} \phi = \underline{\int_a^b} \phi = \alpha(d - c)$. In particular, $\phi \in \mathcal{R}([a, b])$ by definition.

ii. It is clear from last paragraph of (i) that $\int_a^b \phi := \overline{\int_a^b} \phi = \underline{\int_a^b} = \alpha(d - c)$.

Remark. For part (i), as suggested by several of you, alternatively, one can consider any partition $P := \{x_i\}_{i=1}^k$ with $\max_{i=1}^k |x_i - x_{i-1}|$ small enough and then split the sum $\sum_{i=1}^k \omega_i(f, P)\Delta x_i$ into the form

$$\sum_{i=1}^k \omega_i(f, P)\Delta x_i = \sum_{i;[x_{i-1}, x_i] \cap [c,d] = \phi} \omega_i(f, P)\Delta x_i + \sum_{i;[x_{i-1}, x_i] \cap [c,d] \neq \phi} \omega_i(f, P)\Delta x_i$$

You are highly encouraged to try this approach if you have not.