1 (P. 170 Q2)**.** Let $f(x) := x^{1/3}$ for all $x \in \mathbb{R}$. Show that $f : \mathbb{R} \to \mathbb{R}$ is not differentiable at $x = 0$.

Solution. By definition, it is equivalent to show that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^{1/3}}{x} = \lim_{x\to 0} \frac{1}{x^{2/3}}$ does not exist. Consider the sequence given by $x_n := 1/n^3$ for all $n \in \mathbb{N}$. Then $x_n \to 0$ clearly. However, $\frac{1}{x_n^{2/3}} = n^2 \to \infty$. It follows from the sequential criteria that the limit does not exist. Therefore, f is not differentiable at $x = 0$.

2 (P. 170 Q4)**.** Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ Show that f is differentiable at $x = 0$ and find $f'(0)$.

Solution. The existence is equivalent to show that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x}$ exists. To proceed we find the explicit limit and claim that $\lim_{x\to 0} \frac{f(x)}{x} = 0$. Let $\epsilon > 0$. Then take $\delta := \epsilon > 0$. Suppose $x \in \mathbb{R}$ such that $0 < |x - 0| < \delta$. Then

$$\left|\frac{f(x)}{x} - 0\right| = \left|\frac{f(x)}{x}\right| = \frac{|f(x)|}{|x|} \overset{(*)}{\leq} \frac{|x^2|}{|x|} = |x| \leq \delta = \epsilon$$

where $(*)$ follows as $|f(x)| \leq |x^2|$ regardless of $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$, which follows clear from the definition of f. By the definition of limits, $f'(0) = 0$. In particular, f is differentiable at 0.

Common Mistake. I do not accept solutions similar to the following: it is not appropriate to consider x to lie in certain subset before taking limits as you are taking limit with respect to the *domain of f*. To play safe, use $\epsilon - \delta$ definition or limit theorems.

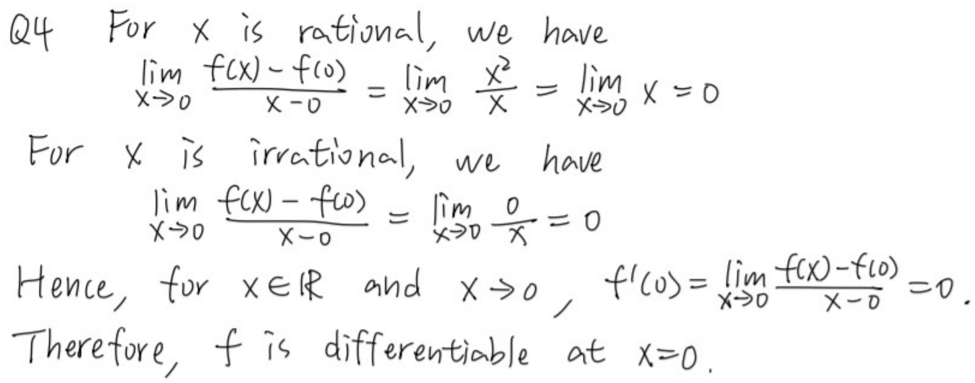

Figure 1: Avoid the above in the future, especially the first two lines

3 (P. 170 Q10)**.** Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) := \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$ Show that

a. g is differentiable on $\mathbb{R}$

b. g' is unbounded on $[-1, 1]$.

Solution.

a. Note that $g(x) = x^2 \sin(1/x^2)$ for all $x \in (0, \infty)$. It follows from the chain rule, product rule, quotient rule and differentiability of trigonometry and polynomial functions on $(0, \infty)$ that g is differentiable on $(0, \infty)$. By a similar argument, g is differentiable on $(-\infty, 0)$. It remains to show that g is differentiable at $x = 0$. To this end, it is equivalent to show the existence of $\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} x \sin(1/x^2)$. Note that $|x \sin 1/x^2| \leq |x|$ for all $x \neq 0$. By the squeeze theorem, it follows that $\lim_{x\to 0} x \sin 1/x^2 = 0$ and so $g'(0) = 0$ and g is differentiable at 0.

b. By part a, together with the chain and product rules, we have that

$$g'(x) := \begin{cases} 2x \sin(1/x^2) - \frac{2}{x} \cos(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Consider the non-negative sequence (x_n) given by $\dfrac{1}{x_n^2} = 2n\pi$ for all $n \in \mathbb{N}$. Note that $\lim x_n \to 0$ clearly and $g'(x_n) = -2\sqrt{2n\pi}$ for all $n \in \mathbb{N}$. It follows that $g'(x_n) \to -\infty$. Hence, g' is unbounded on any neighborhood (open interval) containing 0 (consider $r > 0$ and $M > 0$ and choose n large enough such that $x_n \in (-r, r)$ and $g'(x_n) < -M$). In particular g' is unbounded on $[-1, 1]$