1. Discuss the convergence and the uniform convergence of the series $\sum f_n$ where $f_n(x)$ is given by

i. $f_n(x) := \frac{x^n}{x^n+1}$ for $x \geq 0$

ii. $f_n(x) := \frac{(-1)^n}{n+x}$ for $x \geq 0$

Solution.

i. We first claim that f_n converges point-wise precisely on $[0,1)$. When $x \geq 1$, $f_n(x) = \frac{x^n}{x^n+1} \geq \frac{x^n}{x^n+x^n} = \frac{1}{2}$ for all $n \in \mathbb{N}$. It follows clearly that $\sum_n f_n(x)$ diverges. It remains to consider the convergence of $\sum f_n$ on $[0,1)$. Now suppose $x \in [0,1)$. Then $|f_n(x)| \leq x^n$. It follows from the convergence of geometric series and the comparison test that $\sum f_n(x)$ converges. Hence, $\sum f_n$ converges point-wise on $[0,1)$ but diverges elsewhere.
Next, we claim that f_n is not uniformly converging on its domain of convergence, that is, on $[0,1)$. We proceed by contradiction. Suppose $(\sum f_n)$ converges uniformly on $[0,1)$. Then f_n converges to 0 uniformly. This is because we have $f_n = s_n - s_{n-1}$ where $s_n := \sum_{k=1}^n f_k$ for all $n \in \mathbb{N}$. It follows from the Cauchy criteria (on the uniform convergence of (s_n)) that f_n converges to 0 uniformly. However, for all $n \in \mathbb{N}$, take $x_n := (1/2)^{1/n} \in [0,1)$. It follows that we have $f_n(x_n) = \frac{1/2}{1/2+1} = \frac{1}{3}$. It follows from definitions that f_n does not converge uniformly to 0 and so contradiction arises.

ii. Write $a_n(x) := \frac{1}{n+x}$ and so $f_n(x) = (-1)^n a_n(x)$ for all $x \geq 0$ and $n \in \mathbb{N}$. We claim that f_n converges point-wise on $x \geq 0$. Fix $x \geq 0$ then $a_n(x) := \frac{1}{n+x}$ is decreasing with $\lim a_n(x) = 0$. By the alternating series test, $\sum (-1)^n a_n(x) = \sum f_n(x)$ converges.
Next, we claim that f_n converges uniformly on $[0,\infty)$. To this end, observe that for all $n \in \mathbb{N}$ and $x \geq 0$, we have
$$0 \leq a_n(x) - a_{n+1}(x) = \frac{1}{n+x} - \frac{1}{n+1+x} = \frac{1}{(n+x)(n+1+x)} \leq \frac{1}{n^2}$$
Hence, for all odd $n \in \mathbb{N}$ and $p \geq 1 \in \mathbb{N}$ we have
$$0 \leq \sum_{k=n+1}^{n+p} f_k(x) = \sum_{i=1}^{p/2} a_{n+i}(x) - a_{n+i+1}(x) \leq \sum_{i=1}^{p/2} \frac{1}{n+i} = \sum_{k=n+1}^{n+p/2} \frac{1}{k^2}$$
if p is even; and we have
$$0 \leq \sum_{k=n+1}^{n+p} f_k(x) = \sum_{i=1}^{(p-1)/2} a_{n+i}(x) - a_{n+i+1}(x) + a_{n+p}(x) \leq \sum_{k=n+1}^{n+(p-1)/2} \frac{1}{k^2} + \frac{1}{n}$$
Combining the two, we have
$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| \leq \sum_{k=n+1}^{n+p} \frac{1}{k^2} + \frac{1}{n}$$
for all odd $n \in \mathbb{N}$ and $p \in \mathbb{N}$. We clearly have similar inequalities when n is even. Hence by the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^2}$, it is clear that the Cauchy condition is valid for $\sum f_k$ and so $\sum f_k$ converges uniformly on $x \geq 0$.

Remark. For Q1i, many of you also discussed uniform convergence of the series on compact sets, which is totally relevant and can reward you marks.

2. Let (c_n) be a decreasing sequence of positive numbers. Suppose $\sum c_n \sin nx$ converges uniformly. Show that $\lim nc_n = 0$.

Solution. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|\sum_{k=n}^{2n} c_k \sin(kx)\right| < \epsilon$ for all $x \in \mathbb{R}$ by applying Cauchy criteria on the uniform convergence in question. For all $n \geq N$, we take $x_n := \frac{\pi}{4n}$. Then $nx_n = \pi/4$ and $2nx_n = \pi/2$. It follows that $kx_n \in [\pi/4, \pi/2]$ for all $k \in [n, 2n] \cap \mathbb{N}$. Hence, $\sin(kx_n) \geq \sin(\pi/4) \geq \sqrt{2}/2$ for all $n \geq N$ and $k \in [n, 2n] \cap \mathbb{N}$. It follows that we have for all $n \geq N$ that
$$\epsilon > \left| \sum_{k=n}^{2n} c_k \sin(kx) \right| = \sum_{k=n}^{2n} c_k \sin(kx) \geq \frac{\sqrt{2}}{2} \sum_{k=n}^{2n} c_k \overset{(c_k)\downarrow}{\geq} \frac{\sqrt{2}}{2} \sum_{k=n}^{2n} c_{2n} \geq \frac{\sqrt{2}}{2} nc_{2n}$$
Hence, for all $m \geq 2N+1$, $\lfloor m/2 \rfloor \geq N$. It follows that $\epsilon > \frac{\sqrt{2}}{2} \lfloor m/2 \rfloor c_{2\lfloor m/2 \rfloor} \geq \frac{\sqrt{2}}{8}(4\lfloor \frac{m}{2} \rfloor)c_m \geq \frac{\sqrt{2}}{8} mc_m$. It follows that $\lim nc_n = 0$.

Remark. After proving $\lim 2nc_{2n} = 0$, one can also consider $(2n+1)c_{2n+1} \leq (2n+1)c_{2n} \leq 2nc_{2n} + c_{2n} \leq 2nc_{2n} + 2nc_{2n} \to 0$ as $n \to \infty$. Combining $2nc_{2n} \to 0$ and $(2n+1)c_{2n+1} \to 0$, we have $nc_n \to 0$.

3. Find a series expansion for $f(x) := \int_0^x e^{-t^2} dt$ for $x \in \mathbb{R}$.

Solution. First note that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ for all $t \in \mathbb{R}$ is the power series expansion for $t \mapsto e^t$. Note that the power series converges everywhere on $\mathbb{R}$ and so converges on every compact intervals (on $[-\eta, \eta]$ where $\eta > 0$ in particular). Hence, the (power) series expansion $e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$ also converges uniformly for all compact set. In particular, it converges uniformly on $[0, x]$ for all $x \in \mathbb{R}$. Hence by limit-integral exchange theorems, we have

$$f(x) := \int_0^x e^{-t^2} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{x^{2n+1}}{n!}$$

for all $x \in \mathbb{R}$ to be the required series expansion.

Common Mistake. One **must** verify every interchange of integrals and limits (even in future courses). Hence you must state the uniform convergence for the power series of e^t on *compact* sets. Note the the power series is NOT uniformly converging on $\mathbb{R}$ (try to prove yourself!). A number of you wrote too few for this question.