MATH 2068: Honours Mathematical Analysis II: Home Test 2
5:00 pm, 08 April 2022

Important Notice:

♣ The answer paper **must be submitted before 09 April 2022 at 5:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ **The answer paper must include your name and student ID.**

Answer ALL Questions

1. **(20 points)** Let f be a real valued function defined on $[0, +\infty)$. Prove or disprove the following statements:

   (i) If $\int_0^\infty f(x)dx$ and $\lim_{x\to+\infty} f(x)$ both exist, then $\lim_{x\to+\infty} f(x) = 0$

   (ii) If $\int_0^\infty f(x)dx$ exists, then $\lim_{x\to+\infty} f(x)$ exists.

2. **(30 points)**
Recall that a function g is called a C-Lipschitz function for some $C > 0$ if $|g(x)-g(y)| \leq C|x-y|$ for all x, y in its domain.
Now let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded function and $M := \sup_{x\in\mathbb{R}} |f(x)|$. For each $\lambda > 0$ and $x \in \mathbb{R}$, we put

$$\psi_\lambda(x) := \inf\{g(x) : \ g \text{ is a } \lambda\text{-Lipschitz function on } \mathbb{R} \text{ and } g \geq f \text{ on } \mathbb{R}\ \};$$

and $\psi_0(x) = M$ for all $x \in \mathbb{R}$.
Suppose that for each $t > 0$, there is $\lambda > 0$ such that $\psi_\lambda(x) - f(x) < t$ for all $x \in \mathbb{R}$.
For each $t > 0$, set

$$\tau(t) := \inf\{\lambda > 0 : \psi_\lambda(x) - f(x) < t; \forall x \in \mathbb{R}\}$$

and

$$\varphi(x) := \int_0^1 \psi_{\tau(t)}(x)dt \tag{1}$$

for $x \in \mathbb{R}$.

(i) Show that for each $\lambda > 0$, ψ_λ is a λ-Lipschitz function on $\mathbb{R}$.

(ii) Show that the improper integral in Eq(1) exists for all $x \in \mathbb{R}$, that is, the function $t \in [c, 1] \mapsto \psi_{\tau(t)}(x)$ is Riemann integrable for all $c \in (0, 1]$ and $\lim_{c\to 0+} \int_c^1 \psi_{\tau(t)}(x)dt$ exists.

(iii) Show that the function φ is a bounded uniformly continuous function on $\mathbb{R}$.

$* * *$ **END OF PAPER** $* * *$