MATH 2068: Honours Mathematical Analysis II: Home Test 1
5:00 pm, 04 Mar 2022

Important Notice:

♣ The answer paper **must be submitted before 05 Mar 2022 at 5:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ **The answer paper must include your name and student ID.**

Answer ALL Questions

1. **(25 points)**

   Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Let $a, b \in \mathbb{R}$ with $a < b$.

   (i) Prove or disprove the following statement: for any $c \in \mathbb{R}$, there are numbers x_1, x_2 with $x_1 < c < x_2$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$.

   (ii) Suppose that $f'(a) < d < f'(b)$. Show that there is a point $c \in (a, b)$ such that $f'(c) = d$.

   (iii) Prove or disprove the following statement: if f' is injective, then f' is strictly monotone.

2. **(25 points)**

(i) Let f be a non-constant continuous function defined on $[a, b]$ such that $f(a) = f(b) = 0$. Suppose that f' exists and is bounded on (a, b). Put $M := \sup\{|f'(x)| : x \in (a, b)\}$. Show that $f(x') < M(x' - a)$ for some $x' \in [a, \frac{a+b}{2}]$ or $f(x'') < M(b - x'')$ for some $x'' \in [\frac{a+b}{2}, b]$.

(ii) For each subset A of $\mathbb{R}$, we put $I_A(x) := 1$ whenever $x \in A$; otherwise, $I_A(x) := 0$. Let E be a vector space given by

$$\{h : [a, b] \longrightarrow \mathbb{R} : \ h \text{ is bounded and has at most finitely many discontinuous points}\}.$$

Let $\mu : E \longrightarrow \mathbb{R}$ be a linear function which satisfies the following conditions:

(α) $\mu(h) \geq 0$ for any $h \in E$ with $h \geq 0$, i,e. $h(x) \geq 0$ for all $x \in [a, b]$;

(β) $\mu(\mathbf{1}) = 1$ where $\mathbf{1}(x) = 1$ for all $x \in [a, b]$;

(γ) for any partition $a = x_0 < x_1 < \cdots < x_n = b$ and for any $h \in E$, we have $\mu(h) = \sum_{k=1}^{n} \mu(h \cdot I_{[x_{k-1}, x_k]})$, where $h \cdot I_{[x_{k-1}, x_k]}$ denotes the usual product of functions.

Let f be the function given as in Part (i). Show that there is a point $\xi \in (a, b)$ such that

$$\frac{1}{b - a}\mu(f) < |f'(\xi)|.$$

$* * *$ **END OF PAPER** $* * *$