THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2018)
HW9 Solution

1. (P.247 Q22)
To show the uniform convergence of f_n to f, note that $f_n(x) - f(x) = (x + \frac{1}{n}) - x = \frac{1}{n}$, and hence $\|f_n - f\|_{\mathbb{R}} = \frac{1}{n} \to 0$ as $n \to \infty$. Therefore, by Lemma 8.1.8 of the textbook, f_n converges uniformly to f on $\mathbb{R}$.

To show f_n^2 does not converge uniformly on $\mathbb{R}$, by Lemma 8.1.10 of the textbook, it suffices to find some $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$, there exists $m, n \geq N$ and $x \in \mathbb{R}$ such that

$$\left|f_n^2(x) - f_m^2(x)\right| \geq \epsilon_0$$

Let $\epsilon_0 = 1$, for all $N \in \mathbb{N}$, chooses $m = 2N, n = N, x = N$, then

$$\begin{aligned}\left|f_n^2(x) - f_m^2(x)\right| &= \left|(x + \frac{1}{n})^2 - (x + \frac{1}{m})^2\right| \\ &= \left|(\frac{2}{n} - \frac{2}{m})x + \frac{1}{n^2} - \frac{1}{m^2}\right| \\ &= \left|(\frac{2}{N} - \frac{2}{2N})N + \frac{1}{N^2} - \frac{1}{4N^2}\right| \\ &= 1 + \frac{3}{4N^2} > 1 = \epsilon_0\end{aligned}$$

Therefore, f_n^2 does not converge uniformly on $\mathbb{R}$.

2. (P.247 Q23) Since f_n, g_n converges uniformly to f, g respectively on A, and that f_n, g_n are bounded for all $n \in \mathbb{N}$, there exists $B, C \in \mathbb{R}$ such that $\|f\|_A \leq B$ and $\|g\|_A \leq C$ (Why?). To show $f_n g_n$ converges uniformly to fg on A, we use the definition of uniform convergence:
Let $0 < \epsilon < 1$ be given, by Lemma 8.1.8, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|f_n - f\|_A < \frac{\epsilon}{2(1+C)}$ and $\|g_n - g\|_A < \frac{\epsilon}{2B+1}$. In particular, $\|g_n\|_A \leq \epsilon + C < 1 + C$

Then for all $x \in A$, $n \geq N$,

$$\begin{aligned}|f_n g_n(x) - fg(x)| &\leq |f(x)||g(x) - g_n(x)| + |g_n(x)||f(x) - f_n(x)| \\ &< B \cdot \frac{\epsilon}{2B+1} + (1+C) \cdot (\frac{\epsilon}{2(1+C)}) \\ &< \epsilon\end{aligned}$$

Therefore, $f_n g_n$ converges uniformly to fg on A.

Remark: Many students use the boundness of each function of the sequence (f_n) (similarly for (g_n)) to argue that there exists $M \in \mathbb{R}$ (independent of n) such that $\|f_n\|_A \leq M$ for all $n \in \mathbb{N}$. This is not true in general (consider $f_n(x) \equiv n$ on $\mathbb{R}$) unless (f_n) converges uniformly to some function on A. One has to use Cauchy criterion to argue the existence of such M.

3. (P.252 Q12) We first show that $f_n(x) = e^{-nx^2}$ converges uniformly to 0 on $[1, 2]$: since $e^{nx^2} \geq nx^2 \geq n$ for all $n \in \mathbb{N}$ and $x \in [1, 2]$, $|f_n(x) - 0| = e^{-nx^2} \leq \dfrac{1}{n}$. Therefore, $\|f_n\|_{[1,2]} \leq \dfrac{1}{n} \to 0$ as $n \to \infty$. Therefore, by Lemma 8.1.8, $f_n(x) = e^{-nx^2}$ converges uniformly to 0 on $[1, 2]$.

   Therefore, by Theorem 8.2.4, $\lim\limits_{n\to\infty} \int_1^2 e^{-nx^2} dx = \int_1^2 0 dx = 0$.

4. (P.252 Q20) A reflection followed by a translation of a typical example would do. Let $n \in \mathbb{N}$, let $f_n : [0, 1) \to \mathbb{R}$ be defined by $f_n(x) := nx + (n - 1)$ if $x \in [1 - \frac{1}{n}, 1)$ and $f_n(x) := 0$ otherwise. It is easy to check that $\{f_n\}$ is decreasing and its pointwise limit is 0 constant function but the convergence is not uniform.