THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2018)
HW6 Solution

1. (P.224 Q15)

   Let $x \in \mathbb{R}$ be fixed. To show g is differentiable at x, it suffices to show that $g|_I$ is differentiable at x for some open interval I containing x.

   Let $x' = x - 3c$, and let $I = (x - c, x + c)$, then for all $y \in I$, $y \geq x - c$, and hence $y - c \geq x - 2c > x$'.

   Therefore, for all $y \in I$, we may write

   $$g(y) = \int_{y-c}^{y+c} f(t)dt = \int_{x'}^{y+c} f(t)dt - \int_{x'}^{y-c} f(t)dt = h(y+c) - h(y-c)$$

   where $h(z) = \int_{x'}^{z} f(t)dt$, defined on $(x', +\infty)$ (which contains I). Since f is continuous on $\mathbb{R}$ (in particular on $[x', +\infty)$), by Fundamental Theorem of Calculus (Theorem 2.1 (ii) of the lecture note), h is differentiable on $(x', +\infty)$ with $h'(z) = f(z)$.

   Therefore, on I, since $g(y) = h(y+c) - h(y-c)$, with the fact that h is differentiable on $(x', +\infty)$, which imply h is differentiable at $y + c$ and $y - c$ for all $y \in I$, $g|_I$ is differentiable at x, with $g'(x) = h'(x+c) - h'(x-c) = f(x+c) - f(x-c)$.

   Remark: Most students can recognise g as the difference of two primitives of f. However, only a few could aware that these primitives are defined on some half-interval only (e.g. $[0, +\infty)$ for $F(z) = \int_0^z f(t)dt$). One has to be careful about the domain of these primitives to argue the differentiability of g; also, some of the "standard calculus facts" involving integrations need careful justifications in this course. For instance, one should avoid the convention $\int_b^a f(t)dt = -\int_a^b f(t)dt$ for $a < b$, since $\int_b^a f(t)dt$ does not make sense in our definition of integral.

2. (P.225 Q16)
   It is a proof by contradiction. Without loss of generality, assume there exists $z \in [0, 1]$ s.t. $f(z) := S > 0$ (the proof still works for $f(z) < 0$). By continuity, z can be assumed neither equal to 0 or 1. Again, by continuity, there exists $\delta > 0$ s.t. $|f(x) - S| < S/2$ for all $x \in V_\delta(z) \subset [0, 1]$, we have $\int_z^{z+\delta} f > 0$.

   $$0 = \int_0^{z+\delta} f - \int_{z+\delta}^{1} f = \int_0^z f + \int_z^{z+\delta} f - \int_z^1 f + \int_z^{z+\delta} f > 0.$$

   Contradiction arises.

3. (P.225 Q21)

   (a) Since for all $t \in \mathbb{R}$, $(tf \pm g)^2 \geq 0$, by Prop. 1.12 of Lecture note, we have $\int_a^b (tf \pm g)^2 \geq 0$.

(b) For any $t > 0$, expanding $\int_a^b (tf \pm g)^2$, we have

$$\int_a^b (tf \pm g)^2 = \int_a^b (t^2 f^2 \pm 2tfg + g^2)$$

Since $\int_a^b (tf - g)^2 \geq 0$ by (a), we have

$$2t(\pm \int_a^b fg) \leq t^2 \int_a^b f^2 + \int_a^b g^2$$

Since $t > 0$, the above implies

$$2(\pm \int_a^b fg) \leq t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2$$

Therefore, we have

$$2\Big| \int_a^b fg \Big| \leq t \int_a^b f^2 + \frac{1}{t} \int_a^b g^2$$

(c) If $\int_a^b f^2 = 0$, then by the inquality in (b), for all $t > 0$, we have

$$2\Big| \int_a^b fg \Big| \leq \frac{1}{t} \int_a^b g^2$$

Let $t \to 0$, by sandwich theorem, we have $\left| \int_a^b fg \right| = 0$, and hence $\int_a^b fg = 0$.

(d) (i) $\left| \int_a^b fg \right|^2 \leq \left(\int_a^b \left| fg \right| \right)^2$: By Prop. 1.12 (ii), $\left| \int_a^b fg \right| \leq \int_a^b \left| fg \right|$, squaring both sides imply $\left| \int_a^b fg \right|^2 \leq \left(\int_a^b \left| fg \right| \right)^2$.

(ii) $\left(\int_a^b \left| fg \right| \right)^2 \leq \left(\int_a^b f^2 \right) \cdot \left(\int_a^b g^2 \right)$: Replacing f, g by $|f|$ and $|g|$ respectively, we may assume that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in [a, b]$. Hence the desired inequality becomes

$$\left(\int_a^b fg \right)^2 \leq \left(\int_a^b f^2 \right) \cdot \left(\int_a^b g^2 \right)$$

Case I: $\int_a^b f^2 = 0 :$ By (c), $\int_a^b fg = 0$. Therefore,

Case II: $\int_a^b g^2 = 0 :$ By (c) , with the interchange of the roles of f and g, $\int_a^b fg = 0$. Therefore,

$$\left(\int_a^b fg \right)^2 = 0 = \left(\int_a^b f^2 \right) \cdot \left(\int_a^b g^2 \right)$$

Case III: $\int_a^b f^2 \neq 0$ and $\int_a^b g^2 \neq 0 :$ Apply the inequality in (b) with $t = \dfrac{\sqrt{\left(\int_a^b g^2 \right)}}{\sqrt{\left(\int_a^b f^2 \right)}} > 0$, we have

$$\begin{aligned} 2 \int_a^b fg &\leq \frac{\sqrt{\left(\int_a^b g^2 \right)}}{\sqrt{\left(\int_a^b f^2 \right)}} \int_a^b f^2 + \frac{\sqrt{\left(\int_a^b f^2 \right)}}{\sqrt{\left(\int_a^b g^2 \right)}} \int_a^b g^2 \\ &= 2\sqrt{\left(\int_a^b f^2 \right)}\sqrt{\left(\int_a^b g^2 \right)} \end{aligned}$$

which implies

$$\left(\int_a^b fg\right)^2 \leq \left(\int_a^b f^2\right) \cdot \left(\int_a^b g^2\right)$$

Remark: some students did not aware the cases which $\int_a^b f^2 = 0$ or $\int_a^b g^2 = 0$, each of which will make the inequality in (b) not applicable.