THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2060B Mathematical Analysis II 2017-2018
HW4 Solution

1. (P.215 Q8)
   Suppose $f(x) \neq 0$, then there exists $c \in [a, b]$ such that $f(c) \neq 0$.
   By continuity of f there exists $\delta > 0$ such that $f(x) > \frac{1}{2}f(c)$ for all $x \in I$, where $I = (c-\delta, c+\delta) \cap [a, b]$
   Since $f(x) \geq 0$ for all $x \in [a, b]$, $\int_a^b f(x) \geq \int_I f(x) > |I| \cdot \frac{1}{2}f(c) > 0$, contradiction arises.

   Remark Some students argue that if $U(f, P) > 0$ for any partition P of $[a, b]$, then $\overline{\int} f > 0$. This argument is false since $\overline{\int} f$ is not $U(f, P)$ for some particular P, but the limit of $U(f, P)$ (as $||P||$ tends to 0). The limit of a sequence of positive number may not necessarily be zero. (Any indicator function at one point defined on an interval can serve as a counter example to this argument)

2. (P.215 Q10)
   Let $h = f - g$. Since f, g are continuous, h is also continuous.
   Suppose $f(x) \neq g(x)$ for all $x \in [a, b]$, then $h(x) \neq 0$ for all $x \in [a, b]$. By continuity of h, either $h(x) > 0$ or $h(x) < 0$ for all $x \in [a, b]$.
   WLOG, assume $h(x) > 0$ for all $x \in [a, b]$. Then by Q8, $\int_a^b h > 0$, so $\int_a^b f > \int_a^b g$, contradiction.

3. (P.215 Q12)
   Fix $\epsilon > 0$. Note that $g(x)$ is continuous on $[\frac{\epsilon}{4}, 1]$, so $g(x)$ is integrable on $[\frac{\epsilon}{4}, 1]$.
   Hence, there exists a partition $P := (x_1 = \frac{\epsilon}{4}, x_2, x_3, ..., x_n = 1)$ on $[\frac{\epsilon}{4}, 1]$ such that $U(g|_{[\frac{\epsilon}{4},1]}, P) - L(g|_{[\frac{\epsilon}{4},1]}, P) < \frac{\epsilon}{2}$.
   Consider the partition $P' := (x_0 = 0, x_1 = \frac{\epsilon}{4}, x_2, x_3, ..., x_n = 1)$ on $[0, 1]$.
   Since $-1 \leq g(x) \leq 1$ on $[x_0, x_1]$, we have
   $U(g, P') - L(g, P') \leq (x_1 - x_0) \cdot [1 - (-1)] + [U(g|_{[\frac{\epsilon}{4},1]}, P) - L(g|_{[\frac{\epsilon}{4},1]}, P)] \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$
   Since ϵ can be chosen arbitrarily small, we conclude that g is integrable on $[0, 1]$

   Remark Some students argue that for any partition P of $[0, \frac{\epsilon}{4}]$, $U(g|_{[0,\frac{\epsilon}{4}]}, P) - L(g|_{[0,\frac{\epsilon}{4}]}, P) \leq [1 - (-1)] \cdot \frac{\epsilon}{4} < \epsilon$, so g is integrable on $[0, \frac{\epsilon}{4}]$. This is a bogus argument. To show g is integrable on $[0, \frac{\epsilon}{4}]$, one needs to show for any ϵ_2 **independent of** ϵ, that there exists a partition P of $[0, \frac{\epsilon}{4}]$ such that $U(g|_{[0,\frac{\epsilon}{4}]}, P) - L(g|_{[0,\frac{\epsilon}{4}]}, P) < \epsilon_2$