THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW1 Solution

1. (P.171 Q4)

   We claim that f is differentiable at 0 with $f'(0) = 0$.

   Proof of claim: Let $\epsilon > 0$ be given, choose $\delta = \epsilon > 0$. Then for all $x \in V_\delta(0) \setminus \{0\}$,

   Case 1: x is rational: then $f(x) = x^2$, and hence\epilson

$$\begin{aligned}|\frac{f(x)-f(0)}{x-0} - 0| &= |\frac{x^2}{x}| \\ &= |x| < \delta = \epsilon\end{aligned}$$

   Case 2: x is irrational: then $f(x) = 0$, and hence

$$\begin{aligned}|\frac{f(x)-f(0)}{x-0} - 0| &= 0 \\ &< \delta = \epsilon\end{aligned}$$

   Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in V_\delta(0) \setminus \{0\}$,

$$|\frac{f(x)-f(0)}{x-0} - 0| < \epsilon$$

   Hence, f is differentiable at 0 with $f'(0) = 0$.

2. (P.171 Q6)
   For showing dependence on n, use f_n instead of f. If $n = 1$, if $x > 0$, we have $f_n'(x) = 1$ and if $x < 0$, we have $f_n'(x) = 0$. If $x = 0$, firstly, if $y > 0$

$$|\frac{f_n(y)-f_n(0)}{y-0}| = |\frac{y}{y}| = 1. \tag{1}$$

   Secondly, if $y < 0$,

$$|\frac{f_n(y)-f_n(0)}{y-0}| = 0 \tag{2}$$

   $f_1'(0)$ does not exist. If $n \geq 2$, if $x > 0$, we have $f_n'(x) = nx^{n-1} = nf_{n-1}$ and if $x < 0$, we have $f_n'(x) = 0$. If $x = 0$, firstly, if $y > 0$,

$$|\frac{f_n(y)-f_n(0)}{y-0}| = |\frac{y^n}{y}| = y^{n-1}. \tag{3}$$

Let $\epsilon > 0$, choose $\delta \in (0, \epsilon^{1/(n-1)})$ s.t. for all $y \in (0, \delta)$, $|\frac{f_n(y) - f_n(0)}{y - 0}| < \epsilon$. Secondly, if $y < 0$,

$$|\frac{f_n(y) - f_n(0)}{y - 0}| = 0. \tag{4}$$

Hence, $f_n'(0) = 0$. f_n' is continuous for $n \geq 2$. And as shown, by $f_n' = f_{n-1}$ for $x > 0$, we have f_n' is differentiable for $n \geq 3$.

3. (P.171 Q10)

For $x \neq 0$, $g(x) = x^2 \sin \frac{1}{x^2}$ is a product of functions which are differentiable at x (where $\sin \frac{1}{x^2}$ is differentiable at x by Theorem 6.16). Therefore, by Theorem 6.12, g is differentiable at x.

For $x = 0$, we claim that g is differentiable at 0 with $g'(0) = 0$.

Proof of claim: Let $\epsilon > 0$ be given, choose $\delta = \epsilon > 0$. Then for all $x \in V_\delta(0) \setminus \{0\}$,

$$\begin{aligned} |\frac{g(x) - g(0)}{x - 0} - 0| &= |\frac{x^2 \sin \frac{1}{x^2}}{x}| \\ &= |x \sin \frac{1}{x^2}| \\ &\leq |x| < \delta = \epsilon \end{aligned}$$

Therefore, g is differentiable at 0 with $g'(0) = 0$.

Hence, g is differentiable for all $x \in \mathbb{R}$.

More explicitly, for $x \neq 0$, Chain rule gives $g'(x) = 2x \sin \frac{1}{x^2} - \frac{2\cos \frac{1}{x^2}}{x}$; for $x = 0$, $g'(0) = 0$ by above.

We also claim that g' is unbounded on $[-1, 1]$: It suffices to show that for any $M > 0$, there exists $x \in (0, 1)$ such that $|g'(x)| \geq M$.

Given any $M > 0$, choose $x \in (0, 1)$ satisfying the following inequalities:

$$\begin{cases} \frac{1}{x} > \frac{M}{2} \\ \cos \frac{1}{x^2} = 1; \quad \sin \frac{1}{x^2} = 0 \end{cases}$$

(for instance, choose $x = \frac{1}{\sqrt{2k\pi}}$, where $k \in \mathbb{N}$ is sufficiently large such that $\sqrt{2k\pi} > \frac{M}{2}$)

Then we estimate $|g'(x)|$:

$$\begin{aligned} |g'(x)| &= |2x \sin \frac{1}{x^2} - \frac{2\cos \frac{1}{x^2}}{x}| \\ &= |0 - 2\sqrt{2k\pi}| \\ &= 2\sqrt{2k\pi} \\ &> 2 \cdot \frac{M}{2} = M \end{aligned}$$

Therefore, g' is unbounded on $[-1, 1]$.