THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Tutorial Note 5
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If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

1.

(a) Consider the subspace Z of X consisting of all elements $x = \alpha x_0$ where α is a scalar. On Z we define a linear functional f by

$$f(x) = f(\alpha x_0) = \alpha.$$

f is bounded and has norm $\|x_0\|^{-1}$ because

$$|f(x)| = |f(\alpha x_0)| = |\alpha| = \|x_0\|^{-1}\|x\|.$$

By Hahn-Banach theorem, f has a linear extension $\widetilde{f}$ from Z to X, with norm $\|x_0\|^{-1}$. We can see that

$$\widetilde{f}(x_0) = f(x_0) = 1.$$

(b) Let $g : \mathbb{K} \to X$ be defined by $g(\alpha) = \alpha x_0$. Let $T = g \circ \widetilde{f}$. Then T is linear and

$$Tx_0 = g(\widetilde{f}(x_0)) = g(1) = x_0 \Longrightarrow \|T\| \geq 1.$$

On the other hand, it can be seen that $\|g\| = \|x_0\|$ and $\|T\| \leq \|g\|\|\widetilde{f}\| = 1$. Therefore, $\|T\| = 1$.

Choose $y \in X \setminus Z$ and we have $Ty = \widetilde{f}(y)x_0 \in Z \Longrightarrow Ty \neq y \Longrightarrow T$ is not identity.

2. For $y \in l^2$, we have

$$\|M_x(y)\|_2 = \left(\sum_{k=1}^{\infty} |x(k)y(k)|^2\right)^{\frac{1}{2}} \leq \sup_{k\in\mathbb{N}} |x(k)| \left(\sum_{k=1}^{\infty} |y(k)|^2\right)^{\frac{1}{2}} = \|x\|_\infty \|y\|_2.$$

Hence $\|M_x\| \leq \|x\|_\infty$. Let (e_k) be the Schauder basis for l^2. Clearly, $M_x(e_k) = x(k)e_k$ for $k \in \mathbb{N}$. Then we have

$$\|M_x\| = \sup_{\|y\|=1} \|M_x(y)\|_2 \geq \sup_{k\in\mathbb{N}} \|M_x(e_k)\|_2 = \sup_{k\in\mathbb{N}} |x(k)| = \|x\|_\infty.$$

It follows that $\|M_x\| = \|x\|_\infty$.

3.

$$\|Tf\|_1 = \int_a^b \left|\int_a^x f(t)\, dt\right| dx \leq \int_a^b \int_a^x |f(t)|\, dt\, dx = \int_a^b \int_t^b |f(t)|\, dx\, dt$$
$$= \int_a^b (b-t)|f(t)|\, dt \leq \int_a^b (b-a)|f(t)|\, dt = (b-a)\|f\|_1 \Longrightarrow \|T\| \leq b-a.$$

Take $f_n(t) = (b-t)^n, a \le t \le b$ and then $\|f_n\|_1 = \displaystyle\int_a^b (b-t)^n\,dt = \frac{(b-a)^{n+1}}{n+1}$ and $Tf_n(x) = \dfrac{(b-a)^{n+1} - (b-x)^{n+1}}{n+1}$. Therefore,

$$\|Tf_n\|_1 = \frac{(b-a)^{n+2}}{n+1} - \frac{(b-a)^{n+2}}{(n+1)(n+2)} = \frac{(b-a)^{n+2}}{n+2}.$$

It follows that

$$\|T\| \ge \frac{\|Tf_n\|_1}{\|f_n\|_1} = \frac{(n+1)(b-a)}{n+2}.$$

Let $n \to \infty$ and we have $\|T\| \ge b-a$. As a conclusion, $\|T\| = b-a$.

Example. Hahn-Banach theorem (extension for linear functions). Let X be a real vector space and p a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies

$$f(x) \le p(x), \quad \forall x \in Z.$$

Then f has a linear extension $\widetilde{f}$ from Z to Z satisfying

$$\widetilde{f}(x) \le p(x), \quad \forall x \in X.$$

We consider $X = (\mathbb{R}^2, \|\cdot\|_2), Z = \{(x, 2x) | x \in \mathbb{R}\}$ and the linear functional $f : Z \to \mathbb{R}$ given by $f(x, 2x) = x$. Let P be the orthogonal projection onto Z:

$$P(x,y) = \left(\frac{x+2y}{5}, \frac{2x+4y}{5}\right).$$

Notice that

$$f(x,y) = x = \langle (x,y), (1,0)\rangle = \langle (x,y), P(1,0)\rangle = \left\langle (x,y), \left(\frac{1}{5}, \frac{2}{5}\right)\right\rangle, \quad (x,y) \in Z.$$

Then we have a Hahn-Banach extension of f given by

$$\widetilde{f}(x,y) = \frac{x+2y}{5}.$$

Example. To illustrate the Hahn-Banach theorem, consider a functional f on the Euclidean plane $Z = \mathbb{R}^2$ defined by $f(x) = \alpha_1 x_1 + \alpha_2 x_2, x = (x_1, x_2)$. Then we have $\|f\|_Z = \sqrt{\alpha_1^2 + \alpha_2^2}$. It has a linear extension $\widetilde{f}$ to $X = \mathbb{R}^3$ defined by

$$\widetilde{f}(x) = \alpha_1 x_1 + \alpha_2 x_2,\ x = (x_1, x_2, x_3).$$

Also, given $x_0 = (a, b, c) \ne (0, 0, 0)$, if we define

$$g(x) = \frac{ax_1 + bx_2 + cx_3}{\sqrt{a^2 + b^2 + c^2}},$$

then we have $\|g\| = 1, \quad g(x_0) = \|x_0\|$.

Example. Show that for any sphere $S(0,r)$ in a normed space X and any point $x_0 \in S(0,r)$ there is a hyperplane H_0 through x_0 such that the ball $\overline{B}(0,r)$ lies entirely in one of the two half planes determined by H_0.

Solution. There exists $\widetilde{f} \in X^*$ such that $\widetilde{f}(x_0) = \|x_0\|, \|\widetilde{f}\| = 1$. Define

$$H_0 = \{x | \widetilde{f}(x) = r\}.$$

Then we have $x_0 \in H_0$. Moreover, for any $x \in \overline{B}(0,r)$, it follows that

$$|\widetilde{f}(x)| \leq \|x\| \leq r.$$