Suggested Solution to Homework 2

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P46, 7. If (X, d) is complete, show that $(X, \tilde{d})$, where $\tilde{d} = d/(1 + d)$, is complete.

Proof. Let (X, d) be a complete metric space. Then, for $\tilde{d} = d/(1 + d)$, it is clear that $\tilde{d}$ is nonnegative. Moreover, $\tilde{d}(x, y) = \dfrac{d(x, y)}{1 + d(x, y)} = 0$ if and only if $d(x, y) = 0$, that is $x = y$, since d is a metric. Now, we show that $\tilde{d}$ satisfies the triangle inequality, i.e.

$$\tilde{d}(x, y) \leq \tilde{d}(x, z) + \tilde{d}(z, y), \quad \forall x, y, z \in X.$$

Since d is a metric, then $d(x, y) \leq d(x, z) + d(z, y)$. Note that the function $f(t) = \dfrac{t}{1 + t}$ is increasing on $[0, \infty)$. Therefore,

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(x, z)}{1 + d(x, z)} = \tilde{d}(x, z) + \tilde{d}(z, y).$$

It suffices to show that $(X, \tilde{d})$ is complete. Let (x_n) is a Cauchy sequence in $(X, \tilde{d})$. Then, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. for all $m, n > N$,

$$\tilde{d}(x_n, x_m) = \frac{d(x_n, x_m)}{1 + d(x_n, x_m)} < \frac{\epsilon}{1 + \epsilon}, \qquad (\star)$$

which implies

$$d(x_n, x_m) < \epsilon.$$

Thus, (x_n) is a Cauchy sequence in (X, d).
By the completeness of (X, d), there exists a $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. That is, $\exists N' \in \mathbb{N}$ s.t. for all $m > N'$, $d(x_m, x) < \epsilon$.
Therefore, for all $n, m > \max\{N, N'\}$,

$$\tilde{d}(x_n, x) \leq \tilde{d}(x_n, x_m) + \tilde{d}(x_m, x) < \frac{\epsilon}{1 + \epsilon} + \epsilon < 2\epsilon.$$

So, $\tilde{d}(x_n, x) \to 0$ as $n \to +\infty$. We conclude that $(X, \tilde{d})$ is complete. □

P46, 8. Show that in Prob. 7, completeness of $(X, \tilde{d})$ implies completeness of (X, d).

Proof. Asuume $(X, \tilde{d})$ is complete. Let (x_n) by a Cauchy sequence in (X, d). Then $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n, m > N$ $d(x_n, x_m) < \epsilon$. It yields that

$$\tilde{d}(x_n, x_m) = \frac{d(x_n, x_m)}{1 + d(x_n, x_m)} < \frac{\epsilon}{1 + \epsilon} < \epsilon.$$

Thus, (x_n) is a Cauchy sequence in $(X, \tilde{d})$.
By the completeness of $\tilde{d}$, there exists a $x \in X$ such that $\tilde{d}(x_n, x) \to 0$ as $n \to +\infty$. That is, $\exists N' \in \mathbb{N}$ s.t. $\forall m > N'$, $\tilde{d}(x_m, x) < \dfrac{\epsilon}{1 + \epsilon}$. Therefore, for all $n, m > \max\{N, N'\}$,

$$d(x_n, x) < d(x_n, x_m) + d(x_m, x) = \frac{\tilde{d}(x_n, x_m)}{1 - \tilde{d}(x_n, x_m)} + \frac{\tilde{d}(x_m, x)}{1 - \tilde{d}(x_m, x)} < 2\frac{\frac{\epsilon}{1 + \epsilon}}{1 - \frac{\epsilon}{1 + \epsilon}} = 2\epsilon.$$

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So, $d(x_n, x) \to 0$ as $n \to +\infty$, which implies the completeness of (X, d). $\square$

P46, 14 Does

$$d(x,y) = \int_a^b |x(t) - y(t)| dt$$

define a metric or pseudometric on X if X is (i) the set of all real-valued continous functions on $[a, b]$, (ii) the set of all real-valued Riemann integrable functions on $[a, b]$?

Proof. For $d(x, y) = \int_a^b |x(t) - y(t)|dt$, it follws from the properties of Riemann integral that, whether X be the set of all real-valued contions or Riemann integrable functions on $[a, b]$,

(a) $d(x,y) = \int_a^b |x(t) - y(t)|dt \geq 0,\ d(x,x) = \int_a^b |x(t) - x(t)|dt = 0, \quad \forall x, y \in X;$

(b) $d(x,y) = \int_a^b |x(t) - y(t)|dt = \int_a^b |y(t) - x(t)|dt = d(y,x), \quad \forall x, y \in X;$

(c) $d(x,y) = \int_a^b |x(t) - y(t)|dt \leq \int_a^b |x(t) - z(t)|dt + \int_a^b |z(t) - y(t)|dt = d(x,z) + d(z,y), \quad \forall x, y, z \in X.$

However,

(i) If $X = C[a, b]$ be the set of all real-valued continous functions on $[a, b]$, then $d(x, y) = 0$ yields that $x = y$. Indeed, suppose not, that is, there exists at least a point $t_0 \in [a, b]$ such that $x(t_0) \neq y(t_0)$, then there exists an interval $(t_0 - \delta, t_0 + \delta)$ such that $|x(t_0) - y(t_0)| > 0$ since the function $|x(t) - y(t)|$ is continous. Therefore,

$$d(x,y) = \int_0^t |x(t) - y(t)|dt \geq \int_{t_0-\delta}^{t_0+\delta} |x(t) - y(t)|dt > 0,$$

which is a contradiction!

(ii) If $X = R[a, b]$ be the set of all real-valued Riemann integrable functions on $[a, b]$. Then $d(x, y) = 0$ can not imply $x = y$. For example, define

$$x(t) = \begin{cases} 1, x \in [a, \frac{a+b}{2}], \\ 0, x \in (\frac{a+b}{2}, b]. \end{cases} \qquad y(t) = \begin{cases} 1, x \in [a, \frac{a+b}{2}), \\ 0, x \in [\frac{a+b}{2}, b]. \end{cases}$$

It is clear that $x, y \in R[a, b]$ and $d(x, y) = \int_a^b |x(t) - y(t)| = 0$. But $x(\frac{a+b}{2}) = 1$, $y(\frac{a+b}{2}) = 0$. They are not equal at the point $t = \frac{a+b}{2}$.

Therefore, we conclude that

(i) d is a metric on $C[a, b]$.

(ii) d is only a pseduometic on $R[a, b]$.

$\square$