THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2019)
Suggested Solution of Homework 5: Section 3.5: 3, 9

3. Show directly from the definition that the following are not Cauchy sequences. (2 marks each)

(a) $((-1)^n)$, (b) $\left(n + \dfrac{(-1)^n}{n}\right)$, (c) $(\ln n)$

Solution. Recall that (x_n) is said to be a Cauchy sequence if for every $\epsilon > 0$, we can find some $N \in \mathbb{N}$ such that the following holds:

$$|x_n - x_m| < \epsilon \quad \text{for every } n, m \geq N$$

To show that a sequence (x_n) is **not** Cauchy, we need to find some $\epsilon_0 > 0$, so that for each $N \in \mathbb{N}$, there are some $n, m \geq N$ with $|x_n - x_m| \geq \epsilon_0$.

(a) Let $\epsilon_0 = 1$. For each $N \in \mathbb{N}$, we put $n = 2N$ and $m = 2N + 1$. Note that $|(-1)^n - (-1)^m| = |1 - (-1)| = 2 > \epsilon_0$. Therefore, $((-1)^n)$ is not Cauchy.

(b) Let $x_n = n + \frac{(-1)^n}{n}$, and let $\epsilon_0 = 1$. For each $N \in \mathbb{N}$, we put $n = 2N + 2$ and $m = 2N + 1$. Then,

$$\begin{aligned} |x_n - x_m| &= \left|2N + 2 + \frac{1}{2N+2} - \left(2N + 1 - \frac{1}{2N+1}\right)\right| \\ &= 1 + \frac{1}{2N+2} + \frac{1}{2N+1} \\ &> \epsilon_0 \end{aligned}$$

Therefore, (x_n) is not a Cauchy sequence.

(c) Let $x_n = \ln n$, and let $\epsilon_0 = \ln 2$. For each $N \in \mathbb{N}$, we put $n = 2N$ and $m = N$. Then, we have

$$|x_n - x_m| = |\ln(2N) - \ln N| = \ln 2 = \epsilon_0$$

Therefore, (x_n) is not Cauchy.

9. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence. (4 marks)

Solution. Let $n, p \in \mathbb{N}$. Note that

$$\begin{aligned} |x_{n+p} - x_n| &\leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \ldots + |x_{n+1} - x_n| \\ &< r^{n+p-1} + r^{n+p-2} + \ldots + r^n \\ &= r^n(1 + r + \ldots + r^{p-1}) \\ &= r^n \left(\frac{1 - r^p}{1 - r}\right) \\ &< \frac{r^n}{1 - r} \end{aligned}$$

Notice that $\lim_{n\to\infty} r^n = 0$ for $0 < r < 1$. For each $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $r^n < \epsilon(1-r)$ whenever $n \geq N$. Now, we see that $|x_{n+p} - x_n| < \epsilon$ if $n, p \in \mathbb{N}$ with $n \geq N$. Therefore, (x_n) is a Cauchy sequence.

To see that $\lim_{n\to\infty} r^n = 0$. If $0 < r < 1$, then $r = \dfrac{1}{1+b}$ for some $b > 0$. Indeed, $b = \frac{1}{r} - 1$. Note that $(1+b)^n = 1 + nb + \dfrac{n(n-1)}{2}b^2 + \ldots > nb$ for $n \geq 2$. Hence,

$$0 < r^n = \frac{1}{(1+b)^n} < \frac{1}{nb}$$

By Sandwich Theorem, we have $\lim_{n\to\infty} r^n = 0$.