Note on MATH 2050A: 2019

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Recall that a subset A of $\mathbb{R}$ is said to be *compact* if for any sequence (x_n) in A must have a convergent subsequence $(x_{n(k)})$ of (x_n) such that $\lim_{k\to\infty} x_{n(k)} \in A$.

Proposition 1.1 *Let $f : A \to \mathbb{R}$ is a continuous function defined on a compact subset A of $\mathbb{R}$, then the image $f(A)$ is compact.*

Proof: Let (y_n) be a sequence in $f(A)$. Let (x_n) be a sequence in A such that $f(x_n) = y_n$ for all $n = 1, 2, ...$ Since A is compact, there is a convergent subsequence $x_{n(k)}$ of (x_n) such that $c := \lim_{k\to\infty} x_{n(k)} \in A$. Then by the continuity of f at c, we see that

$$\lim_{k\to\infty} y_{n(k)} = \lim_{k\to\infty} f(x_{n(k)}) = f(c) \in f(A)$$

as desired. The proof is finished. □

Definition 1.2 Let A and B be the subsets of $\mathbb{R}$. A bijection $f : A \to B$ is called a *homeomorphism* if f and its inverse f^{-1} both are continuous.
In this case, we say that the sets A and B are *homeomorphic*.

Proposition 1.3 *Let A and B be subsets of $\mathbb{R}$ and let $f : A \to B$ be a continuous bijection. If A is compact, then the inverse $f^{-1} : B \to A$ is continuous. Hence, the function f is a homeomorphism automatically in this case.*

Proof: Fix an element $b \in B$. It needs to show that if a sequence (y_n) in B converges to b, then the sequence $\lim f^{-1}(y_n) = f^{-1}(b)$. Put $x_n := f^{-1}(y_n)$ and $a := f^{-1}(b)$.
The result will be shown by the contradiction. Suppose that $x_n \nrightarrow a$. Then there is $\varepsilon > 0$ and a subsequence $(x_{n(k)})$ of (x_n) such that

$$|x_{n(k)} - a| \geq \varepsilon \quad \text{for all } k = 1, 2.... \tag{1.1}$$

By the compactness of A, there is a convergent subsequence $(x_{k(i)})$ of $(x_{n(k)})$ such that $c := \lim_{i\to\infty} x_{k(i)} \in A$. Then the Eq 1.1 gives us that $|c - a| \geq \varepsilon$ and hence, $c \neq a$. From this, we have $f(c) \neq f(a) = b$ because f is bijective.
On the other hand, since f is continuous at c, we have $\lim_{i\to\infty} y_{k(i)} = \lim_{i\to\infty} f(x_{k(i)}) = f(c)$. This implies that $b = f(c)$ which leads to a contradiction. The proof is finished. □

Remark 1.4 The following example shows that the assumption of compactness in Proposition 1.3 is essential.

Example 1.5 Let $A = [0, 1) \cup [2, 3]$ and $B = [0, 2]$. Define $f(x) = x$ as $x \in [0, 1)$ and $f(x) = x - 1$ as $x \in [2, 3]$. Then the function f is a continuous bijection from A onto B but the inverse function f^{-1} is not continuous at $y = 1$. So, the function is as required.