MATH4010. Solution 1.

7. Sol. Let $f_n : [0,1] \to \mathbb{R}$ be

$$f_n = \chi_{[0,\frac{1}{n}]}. \qquad n = 1,2,3,\dots$$

Then $\|f_n\|_{L^1} = \frac{1}{n}$, $\|f_n\|_{L^\infty} = 0$.

So f_n converges to 0 in $L^1[0,1]$ but not in $L^\infty[0,1]$.

For any sequence $x = \{x_n\}_{n=1}^{\infty}$, $\|x\|_{\ell^1} = \sum_{n=1}^{\infty} |x_n|$. $\|x\|_{\ell^\infty} = \sup_n |x_n|$

Therefore, $\|x\|_{\ell^\infty} \leq \|x\|_{\ell^1}$.

So any sequences converging in ℓ^1 norm also converges in L^∞ norm. $\square$

8. Proof. Obviously $|||x||| \geq 0$, and thus it suffices to prove that ① $|||\lambda x||| = |\lambda|\, |||x|||$. $\forall x \in X$. $\lambda \in \mathbb{C}$

② $|||x+y||| \leq |||x||| + |||y|||$. $\forall x, y \in X$.

① if $\lambda \neq 0$, $|||\lambda x||| = \inf\{r>0;\ \lambda x \in rC\}$

$= \inf\{r>0;\ x \in \frac{r}{|\lambda|} C\}$ (C is balanced)

$= |\lambda| \inf\{\frac{r}{|\lambda|} > 0,\ x \in \frac{r}{|\lambda|} C\} = |\lambda|\, |||x|||$

if $\lambda = 0$. Since C is convex and balanced, we have $0 = \frac{1}{2}x + \frac{1}{2}(-x) \in C$

Therefore $|||\lambda x||| = 0$.

② Suppose that $x, y \in X$, and $r, s > 0$ such that

$$x \in rC,\ y \in sC.$$

$$x+y = (r+s)\left(\frac{r}{r+s}\frac{x}{r} + \frac{s}{r+s}\frac{y}{s}\right)$$

Since $\frac{x}{r}, \frac{y}{s} \in C$ and C is convex, $\frac{r}{r+s}\frac{x}{r} + \frac{s}{r+s}\frac{y}{s} \in C$

Therefore, $x+y \in (r+s)C$.

It follows from the arbitrary of r and s that

$$|||x+y||| \le |||x||| + |||y|||, \quad \forall x, y \in X.$$

$\square$