Riemann Integrable Functions

Recall an important theorem that help us check the Riemann integrablility of a function:

Theorem (c.f. Theorem 2.10 of Lecture Note)**.** *Let f be a bounded function defined on a closed and bounded interval $[a, b]$. f is Riemann integrable over $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that*

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \varepsilon.$$

Using the above theorem, the following fact can be deduced.

Theorem (c.f. Proposition 2.13 of Lecture Note)**.** *Let $f : [a, b] \to \mathbb{R}$ be a function. If f is continuous or monotone, then f is Riemann integrable over $[a, b]$.*

Remark. The function f is automatically bounded if it is continuous or monotone on $[a, b]$.

Example 1. Let f be Riemann integrable over $[a, b]$. Suppose $\bar{f}(x) = f(x)$ for all but finitely many $x \in [a, b]$. Show that $\bar{f}$ is Riemann integrable over $[a, b]$.

Solution. By induction, it suffices to show the case that $\bar{f} = f$ on $[a, b]$ except at $c \in [a, b]$. Suppose $c \in (a, b)$. (The special cases $c = a$ and $c = b$ are left as an exercise.) Let $\varepsilon > 0$. We need to find a partition $P = \{x_1, x_2, ..., x_n\}$ of $[a, b]$ such that

$$\sum_{i=1}^{n} \omega_i(\bar{f}, P)\Delta x_i < \varepsilon.$$

Since $f \in \mathcal{R}[a, b]$, there exists a partition $Q = \{y_1, y_2, ..., y_m\}$ of $[a, b]$ such that

$$\sum_{j=1}^{m} \omega_j(f, Q)\Delta y_j < \frac{\varepsilon}{2}.$$

Choose $u, v \in [a, b]$ such that $a < u < c < v < b$ and $v - u < \dfrac{\varepsilon}{2(M - m)}$, where m and M are lower and upper bounds of $\bar{f}$ respectively. Take $P = Q \cup \{u, v\}$. The indices $i = 1, 2, ..., n$ of the points in the partition P can be divided into the sets

$$I = \Big\{i : [x_{i-1}, x_i] \subseteq [u, v]\Big\}, \quad \text{and} \quad J = \{1, 2, ..., n\} \setminus I.$$

Therefore, we can estimate:

$$\begin{aligned}
\sum_{i=1}^{n} \omega_i(\bar{f}, P)\Delta x_i &= \sum_{i\in I} \omega_i(\bar{f}, P)\Delta x_i + \sum_{i\in J} \omega_i(\bar{f}, P)\Delta x_i \\
&\le (M - m)\sum_{i\in I} \Delta x_i + \sum_{j=1}^{m} \omega_j(f, Q)\Delta y_j \\
&\le (M - m)(v - u) + \sum_{j=1}^{m} \omega_j(f, Q)\Delta y_j \\
&< (M - m) \cdot \frac{\varepsilon}{2(M - m)} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

It follows that $\bar{f}$ is also Riemann integrable over $[a, b]$.

Example 2. Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ be functions. Suppose f is Riemann integrable over $[a, b]$ and g is continuous. Show that $g \circ f$ is Riemann integrable over $[a, b]$.

Solution. Let $\varepsilon > 0$. Notice that g is bounded and uniformly continuous on $[c, d]$. Hence there exist $M > 0$ and $\delta > 0$ such that $|g(x)| \leq M$ for all $x \in [c, d]$ and

$$|g(s) - g(t)| < \frac{\varepsilon}{2(b-a)}, \quad \text{whenever } s, t \in [c, d] \text{ and } |s - t| < \delta.$$

Since $f \in \mathcal{R}[a, b]$, there exists a partition $P = \{x_1, x_2, ..., x_n\}$ of $[a, b]$ such that

$$\sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \frac{\delta\varepsilon}{4M}.$$

The indices $i = 1, 2, ..., n$ of the points in the partition P can be divided into the sets

$$I = \Big\{i : \omega_i(f, P) < \delta\Big\} \quad \text{and} \quad J = \Big\{i : \omega_i(f, P) \geq \delta\Big\}.$$

Then, we estimate the two sums on the right-hand-side below:

$$\sum_{i=1}^{n} \omega_i(g \circ f, P)\Delta x_i = \sum_{i \in I} \omega_i(g \circ f, P)\Delta x_i + \sum_{i \in J} \omega_i(g \circ f, P)\Delta x_i$$

Note that for any $i \in I$, $|f(x) - f(y)| \leq \omega_i(f, P) < \delta$ whenever $x, y \in [x_{i-1}, x_i]$. Hence

$$|g \circ f(x) - g \circ f(y)| = |g(f(x)) - g(f(y))| < \frac{\varepsilon}{2(b-a)}, \quad \forall x, y \in [x_{i-1}, x_i].$$

Therefore the first sum can be estimated by:

$$\sum_{i \in I} \omega_i(g \circ f, P)\Delta x_i \leq \frac{\varepsilon}{2(b-a)} \sum_{i \in I} \Delta x_i \leq \frac{\varepsilon}{2(b-a)} \cdot (b-a) = \frac{\varepsilon}{2} \tag{1}$$

On the other hand, note that if $i \in J$, then

$$\delta \sum_{i \in J} \Delta x_i \leq \sum_{i \in J} \omega_i(f, P)\Delta x_i \leq \sum_{i=1}^{n} \omega_i(f, P)\Delta x_i < \frac{\delta\varepsilon}{4M} \quad \Longrightarrow \quad \sum_{i \in J} \Delta x_i < \frac{\varepsilon}{4M}$$

Therefore the second sum can be estimated by:

$$\sum_{i \in J} \omega_i(g \circ f, P)\Delta x_i \leq 2M \sum_{i \in J} \Delta x_i < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2} \tag{2}$$

Finally, it follows from (1) and (2) that

$$\sum_{i=1}^{n} \omega_i(g \circ f, P)\Delta x_i = \sum_{i \in I} \omega_i(g \circ f, P)\Delta x_i + \sum_{i \in J} \omega_i(g \circ f, P)\Delta x_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $g \circ f$ is Riemann integrable over $[a, b]$.

Some useful properties of Riemann integrable functions are listed below.

Theorem (c.f. Theorem 2.9 & 2.14 of Lecture Note)**.** *Let $f, g \in \mathcal{R}[a, b]$ and let $\alpha \in \mathbb{R}$.*

(a) $f + g \in \mathcal{R}[a, b]$. In this case,

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(b) $\alpha f \in \mathcal{R}[a, b]$. In this case,

$$\int_a^b \alpha f(x)dx = \alpha \int_a^b f(x)dx.$$

(c) If $f \leq g$, i.e., $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

(d) $|f| \in \mathcal{R}[a, b]$. In this case,

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

Remark. (a) and (b) describe the vector space structure of $\mathcal{R}[a, b]$ and the linearity of the integral. (c) describe the order structure of the integral. The converse of (d) does not hold.

Theorem (c.f. Theorem 2.16 of Lecture Note)**.** *If $f, g \in \mathcal{R}[a, b]$, then $f \cdot g \in \mathcal{R}[a, b]$.*

Remark. The theorem tells us that the function space $\mathcal{R}[a, b]$ is not only a vector space, but also an **algebra**. i.e., a ring with scalar multiplication, or a vector space with multiplication.

Theorem (c.f. Theorem 2.15 of Lecture Note)**.** *Let f be a bounded function defined on $[a, b]$ and $a < c < b$. Then $f \in \mathcal{R}[a, b]$ if and only if $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$. In this case,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Remark. From this theorem, we introduce the following notations for $f \in \mathcal{R}[a, b]$:

$$\int_a^a f(x)dx = 0 \quad \text{and} \quad \int_b^a f(x)dx = -\int_a^b f(x)dx$$

Mean Value Theorem for Integrals (c.f. Theorem 2.18 of Lecture Note)**.** *Let f be a continuous function defined on $[a, b]$ and g be non-negative and Riemann integrable over $[a, b]$. Then there exists $c \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

Corollary. *Let f be a continuous function on $[a, b]$. Then there exists $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Proof. Apply the **Mean Value Theorem for Integrals** with $g = 1$. □

Remark. The value on the right-hand side in the above equality represents the average value of f over $[a, b]$.

Example 3. The **Mean Value Theorem for Integrals** does not hold if the assumption that g being non-negative is dropped.

Proof. Consider the functions $f(x) = g(x) = \sin x$ on $[0, 2\pi]$. Then

$$\int_0^{2\pi} f(x)g(x)dx = \int_0^{2\pi} \sin^2 x dx = \int_0^{2\pi} \frac{1 - \cos 2x}{2} dx = \pi.$$

On the other hand, for any $c \in [0, 2\pi]$,

$$f(c) \int_0^{2\pi} g(x)dx = \sin c \cdot \int_0^{2\pi} \sin x dx = 0.$$

This shows that the equality can never be achieved. □