Series of Functions

Similar to series of numbers, corresponding notations for series of functions can be defined.

Definition (c.f. Definition 9.4.1)**.** Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$ and let f be a function defined on A.

- The series $\sum f_n$ is said to *converge (pointwisely)* to f on A if $\sum f_n(x)$ converges to $f(x)$ for each $x \in A$. In this case, we denote

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x) \quad \text{or} \quad f = \sum_{n=1}^{\infty} f_n.$$

- The series $\sum f_n$ is said to *converge absolutely* on A if $\sum f_n(x)$ is absolutely convergent for each $x \in A$.

- The series $\sum f_n$ is said to *converge uniformly* to f on A if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} f_k(x) - f(x) \right| < \varepsilon, \quad \forall n \geq N, \quad \forall x \in A.$$

The series version of the theorems involving interchange of limits are given below. They are in particular useful to deduce some nice result for power series.

Theorem (c.f. Theorem 9.4.2)**.** *Let (f_n) be a sequence of continuous functions defined on $A \subseteq \mathbb{R}$ and let f be a function defined on A. If $\sum f_n$ converges uniformly to f on A, then f is continuous on A.*

Theorem (c.f. Theorem 9.4.3)**.** *Let (f_n) be a sequence of Riemann integrable functions defined on $[a, b]$ and let f be a function defined on $[a, b]$. If $\sum f_n$ converges uniformly to f on $[a, b]$, then f is Riemann integrable over $[a, b]$ and*

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Theorem (c.f. Theorem 9.4.4)**.** *Let (f_n) be a sequence of differentiable functions defined on (a, b). Suppose that there exists a point $c \in (a, b)$ such that $\sum f_n(c)$ is convergent and $\sum f_n'$ converges uniformly on (a, b). Then $\sum f_n$ converges uniformly to a differentiable function f on (a, b) and*

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x), \quad \forall x \in (a, b).$$

Cauchy Criterion (c.f. 9.4.5)**.** *Let (f_n) be a sequence of functions defined on A. The series $\sum f_n$ is uniformly convergent on $A \subseteq \mathbb{R}$ if and only if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \varepsilon, \quad \forall n \geq N, \quad \forall p \in \mathbb{N}, \quad \forall x \in A.$$

A useful test of uniform convergence of series of functions is given below:

Weierstrass M-Test (c.f. 9.4.6)**.** *Let (f_n) be a sequence of real-valued functions defined on $A \subseteq \mathbb{R}$ and (M_n) be a sequence of positive real numbers such that $|f_n(x)| \le M_n$ for all $x \in A$ and $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on A.*

Example 1 (c.f. Section 9.4, Ex.1)**.** Determine the convergence of the following series of functions on the given domain of x.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, $x \in \mathbb{R}$. (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2x^2}$, $x \neq 0$. (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{x^n+1}$, $x \ge 0$.

Solution. We will apply different tests to determine the convergence of the given series.

(a) We apply the **Weierstrass M-Test** here. Notice that

$$|f_n(x)| = \frac{|\cos nx|}{n^2} \le \frac{1}{n^2}, \quad \forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}.$$

Since the 2-series $\sum 1/n^2$ is convergent, the series is uniformly convergent on $\mathbb{R}$.

(b) The pointwise convergence is easy. Notice that

$$\sum_{n=1}^{\infty} f_n(x) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n^2x^2} = \lim_{N\to\infty} \frac{1}{x^2} \sum_{n=1}^{N} \frac{1}{n^2} = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6x^2}, \quad \forall x \neq 0.$$

For uniform convergence, notice that if we consider $n = N$, $p = 1$ and $x = 1/(n+1)$,

$$|f_{n+1}(x) + \cdots + f_{n+p}(x)| = \frac{1}{(n+1)^2} \cdot (n+1)^2 = 1 > 0.$$

Therefore the **Cauchy Criterion** implies that the series is not uniformly convergent on $\mathbb{R} \setminus \{0\}$. However, for any fixed $a > 0$, the **Weierstrass M-Test** shows that the series converges uniformly on $A = (-\infty, -a] \cup [a, \infty)$:

$$|f_n(x)| = \frac{1}{n^2x^2} \le \frac{1}{n^2a^2}, \quad \forall n \in \mathbb{N}, \quad \forall x \in A.$$

(c) For $0 \le x \le 1$, the series is divergent by the n**-th Term Test**:

$$\lim_{n\to\infty} \frac{1}{x^n+1} = 1, \quad \text{if } x \in [0,1); \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{x^n+1} = \frac{1}{2}, \quad \text{if } x = 1.$$

For $x > 1$, we have

$$|f_n(x)| = \frac{1}{x^n+1} \le \frac{1}{x^n} = \left(\frac{1}{x}\right)^n, \quad \forall n, \in \mathbb{N}, \quad \forall x > 1.$$

The **Comparison Test** implies that the series is (absolutely) convergent on $(1, \infty)$. Similar to the above example, we can show that the series does not convergent uniformly on $(1, \infty)$. On the other hand, we can verify by the **Weierstrass M-Test** that for any fixed $a > 1$, the series converges uniformly on $[a, \infty)$.

Power Series

Definition (c.f. Definition 9.4.7)**.** Let (a_n) be a sequence of real numbers and $c \in \mathbb{R}$. A (formal) *power series around* c is in the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n.$$

Denote $\operatorname{dom}(f)$ as the set of $x \in \mathbb{R}$ for which the series is convergent.

Example 2 (c.f. Section 9.4, Ex.6(e))**.** Find $\operatorname{dom}(f)$, where f is the following power series.

$$f(x) = \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n$$

Solution. We always have $0 \in \operatorname{dom}(f)$. For $x \neq 0$, we wish to apply the **Ratio Test**. Notice that for each $n \in \mathbb{N}$,

$$\left| \frac{[(n+1)!]^2 x^{n+1}/(2n+2)!}{(n!)^2 x^n/(2n)!} \right| = |x| \cdot \frac{(n+1)^2}{(2n+2)(2n+1)} = |x| \cdot \frac{n+1}{2(2n+1)}.$$

It follows that

$$\lim_{n\to\infty} \left| \frac{[(n+1)!]^2 x^{n+1}/(2n+2)!}{(n!)^2 x^n/(2n)!} \right| = \lim_{n\to\infty} |x| \cdot \frac{n+1}{2(2n+1)} = \frac{|x|}{4}.$$

We claim that $\operatorname{dom}(f) = (-4, 4)$ by the following arguments:

- If $|x| < 4$, choose r such that $|x|/4 < r < 1$. Then there exists $K \in \mathbb{N}$ such that

  $$\left| \frac{[(n+1)!]^2 x^{n+1}/(2n+2)!}{(n!)^2 x^n/(2n)!} \right| \leq r, \quad \forall n \geq K.$$

  The **Ratio Test** implies that $f(x)$ is (absolutely) convergent.

- If $|x| > 4$, choose r such that $1 < r < |x|/4$. Then there exists $K \in \mathbb{N}$ such that

  $$\left| \frac{[(n+1)!]^2 x^{n+1}/(2n+2)!}{(n!)^2 x^n/(2n)!} \right| \geq r > 1, \quad \forall n \geq K.$$

  The **Ratio Test** implies that $f(x)$ is divergent.

- If $x = \pm 4$, notice that

  $$f(4) = \sum_{n=0}^{\infty} \frac{(n!)^2 \cdot 4^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(2^n \cdot n!)^2}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

  $$f(-4) = \sum_{n=0}^{\infty} \frac{(n!)^2 \cdot (-4)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2^n \cdot n!)^2}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

  The absolute value of each term of the series is greater than 1. The n**-th Term Test** implies that $f(\pm 4)$ are divergent.

The following theorem describe a remarkable property of power series.

Theorem. *Suppose a power series f around 0 is convergent at some $c \in \mathbb{R}$. Then*

(a) $f(x)$ is absolutely convergent whenever $|x| < |c|$.

(b) f converges uniformly on $[-\eta, \eta]$ whenever $|\eta| < |c|$.

Remark. Notice the following:

- By a translation $x \mapsto x + c$, we can always consider power series around 0.
- For any power series f around 0,
  - $0 \in \mathrm{dom}(f)$.
  - $\mathrm{dom}(f)$ must be an interval. i.e., $\mathrm{dom}(f)$ takes one of the following forms:
  $$\{0\}, \quad (-\infty, \infty), \quad (-r, r), \quad [-r, r], \quad (-r, r], \quad [-r, r).$$
  In this case, $r \in [0, \infty]$ is called the *radius of convergence* of f.
  - The convergence of $f(\pm r)$ is not clear.
  - The uniform convergence of f on $\mathrm{dom}(f)$ is not clear.

The continuity, integrability and differentiability of power series are summerized as follows.

Theorem. *Suppose f is a power series around 0 that converges on $(-r, r)$. i.e.,*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \forall x \in (-r, r).$$

Then:

(a) f is continuous on $(-r, r)$.

(b) The indefinite integral of f can be obtained by integrating f term-by-term. i.e.,

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1} x^n}{n}, \quad \forall x \in (-r, r).$$

(c) The derivative of f can be obtained by differentiating f term-by-term. i.e.,

$$f'(x) = \sum_{n=0}^{\infty} \Big(a_n x^n\Big)' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \forall x \in (-r, r).$$

Corollary. *Suppose f is a power series around 0 that converges on $(-r, r)$. Then f is $\mathcal{C}^\infty$ on $(-r, r)$. i.e., f is k-times differentiable on $(-r, r)$ for all $k \in \mathbb{N}$. Moreover, for each $k \in \mathbb{N}$, the k-th derivative of f is given by*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}, \quad \forall x \in (-r, r).$$