MATH 2060B - HW 8 - Solutions[1]

1 (P.286 Q1f)**.** Discuss the convergence and the uniform convergence of the series of functions $\sum f_n$ where $f_n : [0, \infty) \to \mathbb{R}$ are defined by

$$f_n(x) := \frac{(-1)^n}{n + x}$$

for all $x \in [0, \infty)$ and $n \in \mathbb{N}$

Solution.

Convergence of the series: We claim that the series of functions converges pointwise on $[0, \infty)$. Let $x \in [0, \infty)$. Then $(y_n := f_n(x) = \frac{(-1)^n}{n+x})$ is an alternating sequence of real numbers. Clearly $|y_n|$ is non-negative decreasing and $\lim_n |y_n| = 0$. By the alternating series test, we claim that $\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} y_n$ converges.

Uniform convergence of the series: we claim that the series of functions converges uniformly on $[0, \infty)$.

First notice that for all $x \in [0, \infty)$ and $n \in \mathbb{N}$, we have

$$|f_n(x) + f_{n+1}(x)| = \left| \frac{(-1)^n}{n + x} + \frac{(-1)^{n+1}}{n + 1 + x} \right| = \left| \frac{1}{n + x} - \frac{1}{n + 1 + x} \right| = \frac{1}{(n + x)(n + 1 + x)} \leq \frac{1}{n^2}$$

We proceed to show the uniform convergence by Cauchy Criteria. Let $\epsilon > 0$. Since the p-series $\sum \frac{1}{n^2}$ converges, there exists $N_1 \in \mathbb{N}$ such that $\sum_{k=n}^{m} \frac{1}{k^2} < \epsilon$ for $n, m \geq N_1$. There exists $N_2 \in \mathbb{N}$ $\frac{1}{N_2} < \epsilon$. Now take $N := \max\{N_1, N_2\}$, let $n \geq N$ and $p \in \mathbb{N}$. For simplicity, suppose p is even, then we have for all $x \in [0, \infty)$

$$\begin{aligned}
\left| \sum_{k=n}^{n+p} f_n(x) \right| &= \left| (f_n(x) + f_{n+1}(x)) + \cdots + (f_{n+p-2}(x) + f_{n+p-1}) + f_{n+p(x)} \right| \\
&\leq \frac{1}{n^2} + \frac{1}{(n + 2)^2} + \cdots + \frac{1}{(n + p - 2)^2} + \frac{1}{n + p + x} \\
&\leq \epsilon + \frac{1}{n} \leq 2\epsilon
\end{aligned}$$

We conclude that $\sum f_n(x)$ converges uniformly on $[0, \infty)$

[1] Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

2 (P.286 Q17)**.** Let $x \in \mathbb{R}$. Suppose $|x| < 1$. Show that

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

Solution. First we note the follow standard fact about geometric series: for all $|x| < 1$, we have the power series expansion for the rational function

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

Since $|x| < 1$ would imply $\left|-x^2\right| < 1$, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for all $|x| < 1$.
Second, note that we have the following fact on definite intergral: for all $x \in \mathbb{R}$, we have

$$\int_0^x \frac{1}{1+t^2} dt = \arctan(x)$$

Finally, combining the two, if we suppose $|x| < 1$, we have

$$\arctan(x) = \int_0^x \frac{1}{1+t^2} dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n} dt \overset{(*)}{=} \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

where the equality $(*)$ follows from the last Theorem of Tutorial 11.

3 (P.286 Q19)**.** Find a series expansion for $f(x) := \displaystyle\int_0^x e^{-t^2} dt$ for all $x \in \mathbb{R}$.

Solution. Recall that we have the following power series expansion the the exponential function: for all $x \in \mathbb{R}$, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Hence, for all $t \in \mathbb{R}$ as $-t^2 \in \mathbb{R}$, we have the power series expansion

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Hence, we have

$$f(x) := \int_0^x e^{-t^2} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} dt \overset{(*)}{=} \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n t^{2n}}{n!} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$$

for all $x \in \mathbb{R}$ where the equality $(*)$ follows from the last Theorem of Tutorial 11 as in Question 2.
Comment.

- In Question 2 and 3, the equality $(*)$, that is, the interchange of summation and integral, basically follows from the uniform convergence of the (power) series of function on the domain of integration $(0, x)$.
- One has to specify the domain when considering uniform convergence of a series of functions. Power series converges uniformly on an open interval *compactly contained* in the domain of convergence (that is whose closure is contained in the domain). In general it needs special treatments to see if power series converges uniformly on the whole domain of convergence.