MATH 2060B - HW 6 - Solutions[1]

Questions:

1 (P.246 Q12)**.** Let (f_n) be the sequence of functions defined on $\mathbb{R}$ by $f_n(x) := \frac{nx}{1+n^2x^2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then it is known that (f_n) converges pointwise to 0.

i. Show that the sequence converges uniformly on $[a, \infty)$ if $a > 0$.

ii. Show that the sequence does not converge uniformly on $[0, \infty)$

Solution.

i. We first show that f_n is decreasing on $[\frac{1}{n}, \infty)$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Note that f_n is differentiable on $\mathbb{R}$ and the derivative f_n' is given by for all $x \in \mathbb{R}$

$$f_n'(x) = \frac{n(1+n^2x^2) - 2xn^2(nx))}{(1+n^2x^2)^2} = \frac{n - n^3x^2}{(1+n^2x^2)^2} = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$

If $x \geq \frac{1}{n}$, then $x^2 \geq \frac{1}{n^2}$, which implies $1 - n^2x^2 \leq 0$ and so $f_n'(x) \leq 0$. Hence, $f_n' \leq 0$ on $[\frac{1}{n}, \infty)$. By the Mean Value Theorem, it follows that f_n is decreasing on $[\frac{1}{n}, \infty)$.
Next, since $a > 0$, there exists $N_1 \in \mathbb{N}$ such that $a > \frac{1}{N_1}$ (why does such N_1 exist?). Now we are ready to show the uniform convergence of f_n to 0.
Let $\epsilon > 0$. Then by the point-wise convergence of (f_n) at a there exists $N_2 \in \mathbb{N}$ such that $f_n(a) = |f_n(a) - 0| < \epsilon$ for all $n \geq N_2$. Now take $N \geq N_1, N_2$. Suppose $n \geq N$ and let $x \in [a, \infty)$. Then $x \geq a \geq \frac{1}{N_1} \geq \frac{1}{N} \geq \frac{1}{n}$. Since f_n are decreasing on $[\frac{1}{n}, \infty)$, it follows that

$$|f_n(x) - 0| = f_n(x) \leq f_n(a) = |f_n(a)| \leq \epsilon$$

where the last inequality follows from the choice of N_2. By definition, (f_n) converges uniformly to 0 on $[a, \infty)$.

ii. Since (f_n) converges pointwise to 0 on $[0, \infty) \subset \mathbb{R}$, it suffices to show that (f_n) does not converge uniformly to 0 on $[0, \infty)$.
Take $\epsilon_0 := \frac{1}{2}$. Define $x_n := \frac{1}{n} \in [0, \infty)$ for all $n \in \mathbb{N}$. Then it follows that

$$\|f_n - 0\|_{\infty,[0,\infty)} := \sup_{x \in [0,\infty)} |f_n(x) - 0| \geq |f_n(x_n)| = \frac{nx_n}{1+n^2x_n^2} = \frac{n \cdot \frac{1}{n}}{1+n^2 \cdot \frac{1}{n^2}} = \frac{1}{2} \geq \epsilon_0$$

for all $n \in \mathbb{N}$.
Hence, (f_n) does not converge uniformly to 0 on $[0, \infty)$.

Comment.

i. The above shows a solution with the help of derivatives, which is attempted by some of you. In fact, there exists a more elementary solution by observing that

$$|f_n(x) - 0| = \frac{nx}{1+n^2x^2} \leq \frac{nx}{n^2x^2} \leq \frac{1}{nx} \leq \frac{1}{na}$$

for all $x \geq a > 0$ and $n \in \mathbb{N}$

ii. -

[1] Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

2 (P.246 Q22)**.** Let (f_n) be a sequence of functions defined by $f_n(x) := x + \frac{1}{n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Define $f(x) := x$ for all $x \in \mathbb{R}$.

i. Show that (f_n) converges uniformly to f on $\mathbb{R}$.

ii. Show that (f_n^2) does not converge uniformly on $\mathbb{R}$.

Remark. This shows that uniform convergence may not be preserved by multiplication.

Solution.

i. Note that for all $n \in \mathbb{N}$, we have

$$\|f_n - f\|_{\infty,\mathbb{R}} := \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| x + \frac{1}{n} - x \right| = \frac{1}{n}$$

It follows that $\lim_n \|f_n - f\|_{\infty,\mathbb{R}} = 0$. Hence (f_n) converges to f uniformly on $\mathbb{R}$.

ii. Denote $g_n := f_n^2$ for all $n \in \mathbb{N}$. Then $g_n(x) = (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. It is clear that $\lim_n g_n(x) = x^2$ for all $x \in \mathbb{R}$, so (g_n) converges pointwise to $g(x) := x^2$ on $\mathbb{R}$. It remains to show the (g_n) does not converge to g uniformlly. Note that for all $n \in \mathbb{N}$, we have

$$\|g_n - g\|_{\infty,\mathbb{R}} := \sup_{x\in\mathbb{R}} |g_n(x) - g(x)| \geq |g_n(n) - g(n)| = \left| n^2 + \frac{2n}{n} + \frac{1}{n^2} - n^2 \right| = 2 + \frac{1}{n^2} \geq 2$$

It follows that $\lim_n \|g_n - g\|_{\infty,\mathbb{R}}$ does not converge to 0 and so (g_n) does not converge to g uniformly on $\mathbb{R}$.

Comment.

i. -

ii. It is insufficient to claim that (f_n^2) does not converge uniformly by claiming the it does not converge to $g(x) := x^2$ uniformly *without* first showing that g is the point-wise limit of (f_n^2).

3 (P.246 Q23)**.** Let (f_n), (g_n) be sequences of *bounded* functions on some subset $A \subset \mathbb{R}$ which converge uniformly on A to f, g respectively.
Show that $(f_n g_n)$ converges uniformly on A to fg.

Solution. For a function $h : A \to \mathbb{R}$, denote $\|h\|_{\infty,A} := \sup_{x\in A} |h(x)|$ which may or may not be finite. It follows from the Axiom of Completeness and the boundedness assumption that $\|f_n\|_{\infty,A} < \infty$ and $\|g_n\|_{\infty,A} < \infty$ for all $n \in \mathbb{N}$.
We first show that (f_n) and (g_n) are *uniformly* bounded functions, that is $\sup_n \|f_n\|_{\infty,A} < \infty$ and $\sup_n \|g_n\|_{\infty,A} < \infty$. Without loss of generality, we consider only (f_n).
Since (f_n) converges uniformly to f, it follows that $\lim_n \|f_n - f\|_{\infty,A} = 0$. By definition of sequential limit, there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $\|f_n - f\|_{\infty,A} < 1$. Hence for all $x \in A$, we have

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq \|f - f_N\|_{\infty,A} + \|f_N\|_{\infty,A} = 1 + \|f_N\|_{\infty,A}$$

It follows that $\|f\|_{\infty,A} < \infty$ by taking supremum, implying that f is in fact bounded.
Furthermore, for all $n \geq N$ and $x \in A$, we have

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq \|f_n - f\|_{\infty,A} + \|f\|_{\infty,A} < 1 + \|f\|_{\infty,A}$$

It follows that $\|f_n\|_{\infty,A} \leq 1 + \|f\|_{\infty,A}$ for all $n \geq N$. It is then easy to see that

$$\sup_n \|f_n\|_{\infty,A} \leq \max\{\|f_1\|_{\infty,A}, \cdots, \|f_{N-1}\|_{\infty,A}, 1 + \|f\|_{\infty,A}\} < \infty$$

It follows that (f_n) is uniformly bounded and so is (g_n) by similar argument.
We proceed to show that $(f_n g_n)$ converges to fg.
Write $F := \sup_n \|f_n\|_{\infty,A}$ and $G := \sup_n \|g_n\|_{\infty,A}$. Then for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - g_n(x)f(x) + g_n(x)f(x) - g(x)f(x)| \\
&\leq |g_n(x)||f_n(x) - f(x)| + |f(x)||g_n(x) - g(x)| \\
&\leq G\|f_n - f\|_{\infty,A} + \|f\|_{\infty,A}\|g_n - g\|_{\infty,A}
\end{aligned}$$

It follows that by taking supremum, we have

$$\|f_n g_n - fg\|_{\infty,A} \leq G\|f_n - f\|_{\infty,A} + \|f\|_{\infty,A}\|g_n - g\|_{\infty,A}$$

for all $n \in \mathbb{N}$. It follows from the uniform convergence of (f_n) and (g_n) that $\lim_n \|f_n - f\|_{\infty,A} = 0$ and $\lim_n \|g_n - g\|_{\infty,A} = 0$. Hence, we have $\lim_n \|f_n g_n - fg\|_{\infty,A} = 0$ by Squeeze Theorem. It follows that $(f_n g_n)$ converges to fg uniformly.

(Comments on the next page)

Comment.

1. Please note that *a sequence of bounded functions* (f_n) is different from *a bounded sequence of functions* (f_n). The former means that for all $n \in \mathbb{N}$, f_n is a bounded function while the latter does not make sense at this stage[2]. We would not use the term *a bounded sequence of functions* unless we define it explicitly.

2. If (f_n) is a sequence of bounded functions, it means that $\sup_{x\in A} |f_n(x)|$ (also written as $\|f_n\|_{\infty,A}$) is finite by the Axiom of Completeness, but it DOES NOT mean that $\sup_n \|f_n\|_{\infty,A}$ is finite. If the latter condition is satisfied, the sequence (f_n) is called a sequence of *uniformly bounded* functions. In other words, it is not okay to claim that there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in A$ without proof. Only around 10 of you got things right.

3. Likewise, proofs are required to show that the uniform limits f, g are bounded functions and that does not follow from the definition of (f_n) and (g_n) being sequences of bounded functions.

4. The double-bar notation $\|f\|$ (as in the textbook) refers to the *norm*[3]of the function f, which is a real number. There are different kinds of norms and in this course, we would be only considering the *uniform norm* of a function f, denoted by $\|f\|_{\infty,A}$, where A is the domain of f. At this stage, one can just consider *uniform norm* to be a fancy way of denoting the supremum of the function value over the set A.
   On the other hand, the single-bar notation $|f|$ refers NOT to a real number, but a real-valued function. In particular, it refers to the composition of f and the absolute value function, that is, $|f| := |\cdot| \circ f$. One should be careful about the notations.

5. The solution here uses extensively the characterization that a sequence of functions (f_n) converges uniformly to a function f on some domain $A \subset \mathbb{R}$ if and only if $\lim_n \|f_n - f\|_{\infty,A} = 0$.

6. As opposed to Question 2, this shows that uniform convergence is preserved under pointwise multiplication *if the functions are bounded.*

[2]For a sequence of real numbers (x_n), we can say that it is bounded if the set $\{x_n\}$ is bounded in real numbers: to define boundedness for a sequence of objects, we need to define boundedness of subsets for the underlying set ($\mathbb{R}$ in the previous example). At this stage, it does not make sense to talk about a sequence of functions (f_n) to be bounded because we have not introduced what it means for a subset of real-valued functions to be bounded. In order for us to talk about boundedness of sets of functions, we need some notions of lengths or in general norms and even topologies.

[3]As mentioned in the previous footnote, to talk about boundedness of subsets of functions, we need some notion of lengths. In fact norm is a kind of such notions. With a norm (on a vector space), one can then do analysis in general (vector) spaces that is analogous to analysis on $\mathbb{R}$. For details, please refer to the course *Functional Analysis.*