MATH 2060B - HW 3 - Solutions[1]

1 (P.187 Q5)**.** Let $f(x) := \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ and let $g(x) := \sin(x)$ for all $x \in \mathbb{R}$.

a. Show that $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

b. Show that $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

Solution.

a. For $x \neq 0$, we have $f(x) = x^2 \sin(1/x)$ and $g(x) = \sin(x)$. Let $J = (-r, r)\backslash\{0\}$ (where $r > 0$) be a deleted neighborhood of 0 such that $g \neq 0$ on J. Then we have on J,

$$\left|\frac{f(x)}{g(x)}\right| = \left|x^2 \sin(1/x)\right|\sin(x) \leq \left|\frac{x}{\sin x}\right||x|$$

Note that $\lim_{x\to 0} \left|\frac{x}{\sin x}\right||x| = \lim_{x\to 0} \left|\frac{x}{\sin x}\right| \lim_{x\to 0} |x| = 1 \cdot 0 = 0$. The result then follows from the Sandwich Theorem.

b. For $x \neq 0$, we have $f'(x) = 2x \sin(1/x) - \cos(1/x)$ by the chain rule and $g'(x) = \cos(x)$. Let $J' = (-r', r')\backslash\{0\}$ (where $r' > 0$) be a deleted neighborhood of 0 such that $g' \neq 0$ on J. Then we have on J',

$$\frac{f'(x)}{g'(x)} = \frac{2x \sin(1/x)}{\cos x} - \frac{\cos(1/x)}{\cos x}$$

Now take a sequence (x_n) in J such that $\sin(1/x_n) = 0$, $\cos(1/x_n) = 1$ for all $n \in \mathbb{N}$ and $x_n \to 0$ (how?). Then $\frac{f'(x_n)}{g'(x_n)} = -1/\cos x_n$.
By continuity of cosine function, we have $\lim_n f'(x_n)/g'(x_n) = -1/\cos 0 = -1$. Similarly, take another sequence (y_n) in J such that $\sin(1/x_n) = 0$, $\cos(1/x^n) = -1$ for all $n \in \mathbb{N}$ together with that $x_n \to 0$ (how?). Then $\frac{f'(x_n)}{g'(x_n)} = 1/\cos x_n$. By continuity of cosine, we have $\lim_n f'(y_n)/g'(y_n) = 1/\cos 0 = 1$. By sequential criteria, the limit in question does not exist.

Comment.

1. This exercise shows that the converse of the L'Hospital Rule is not true in general: let $f, g : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$ such that $f(c) = g(c) = 0$. Then the existence of $\lim_{x\to c} f(x)/g(x)$ does not imply the existence of $\lim_{x\to c} f'(x)/g'(x)$.

2. For part b, the existence of the limit is equivalent to considering only $\cos(1/x)$.

[1]Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

2 (P.196 Q10)**.** Let $h(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ for all $x \in \mathbb{R}$.

a. Show that $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

b. Suppose $x \neq 0$. Show that the remainder term obtained by applying the Taylor's Theorem to the points $x, x_0 := 0$ and h as an $n-$ times differentiable function does not converge to 0 as $n \to \infty$

Hint: Try to first show that $\lim_{x\to 0} h(x)/x^k = 0$ for all $k \in \mathbb{N}$ by the L'Hospital Rule. The Leibniz's Rule, or the high-order product rule, may be useful to compute $h^n(x)$ for $x \neq 0$ and $n \in \mathbb{N}$ in the process: let $f, g : I \to \mathbb{R}$ be functions defined on an open interval I and $n \in \mathbb{N}$. The Leibniz's Rule states that if f, g are n-times differentiable at $x \in I$, then the derivative of the product at x can be computed by $(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x) g^k(x)$.

Solution.

a. Following the hint, we first show that $\lim_{x\to 0} h(x)/x^k$ for all $k \in \mathbb{N}$.
We proceed by induction. When $n = 1$: take $f_1(x) := 1/x$ and $g(x) := e^{1/x^2}$. Then f_1, g are differentiable on some deleted neighborhood of 0, say $J := (-r, r)\backslash\{0\}$ where $r > 0$. Furthermore $\lim_{x\to 0} g(x) = \infty$ and $g'(x) = -2x^{-3}e^{1/x^2} \neq 0$ for all $x \in J$. Hence by the L'Hospital Rule, we have
$$\lim_{x\to 0} \frac{h(x)}{x} = \lim_{x\to 0} \frac{f_1(x)}{g(x)} = \lim_{x\to 0} \frac{f_1'(x)}{g'(x)} = \lim_{x\to 0} \frac{-x^{-2}}{-2x^{-3}e^{1/x^2}} = \lim_{x\to 0} \frac{1}{2} x h(x) = 0$$
Set $k \geq 2$. Now suppose $\lim_{x\to 0} h(x)/x^j = 0$ for all $1 \leq j < k$. Take $f_n(x) := 1/x^n$ for $n \in \mathbb{N}$. Then by the L'Hospital Rule, we have
$$\lim_{x\to 0} \frac{h(x)}{x^k} = \lim_{x\to 0} \frac{f_k(x)}{g(x)} = \lim_{x\to 0} \frac{f_k'(x)}{g'(x)} = \lim_{x\to 0} \frac{-kx^{-k-1}}{-2x^{-3}e^{1/x^2}} = \lim_{x\to 0} \frac{k}{2} \frac{h(x)}{x^{k-2}} = \lim_{x\to 0} \frac{k}{2} \frac{h(x)}{x^{k-1}} x = 0$$
where the last equality follows from the induction hypothesis.
Note that since taking limits respect product, sum and scalar multiplications, it follows readily from the claim that $\lim_{x\to 0} h(x)P(1/x) = 0$ for all polynomials P with real coefficients.
Next, we claim that for all $n \in \mathbb{N}$, there exists a polynomial P_n such that $h^{(n)}(x) = P_n(1/x)h(x)$ for all $x \neq 0$. We proceed again by induction:
When $n = 1$, we have $h'(x) = 2x^{-3}e^{-1/x^2} = 2x^{-3}h(x) = P_1(1/x)h(x)$ for all $x \neq 0$ where $P_1(t) := 2t^3$ is a real polynomial.
Let $k \geq 2$. Now suppose for all $j < k$ there exists polynomial P_j such that $h^{(j)}(x) = P_j(1/x)h(x)$ for all $x \neq 0$. Then $h^{(k)}(x) = (h^{(k-1)}(x))' = (P_{k-1}(1/x)h(x))'$ for all $x \neq 0$. By the product rule and chain rule, we have for all $x \neq 0$
$$\begin{aligned} h^{(k)}(x) = (P_{k-1}(1/x)h(x))' &= -1/x^2 P_{k-1}'(1/x)h(x) + P_{k-1}(1/x)h'(x) \\ &= -1/x^2 P_{k-1}'(1/x)h(x) + P_{k-1}(1/x)P_1(1/x)h(x) \\ &= (-1/x^2 P_{k-1}'(1/x) + P_{k-1}(1/x)P_1(1/x))h(x) \end{aligned}$$
which is again the product of a polynomial (with variable $1/x$) and $h(x)$. Hence by induction, for all $n \in \mathbb{N}$, there exists polynomials P_n such that $h^{(n)}(x) = P_n(1/x)h(x)$ for all $x \neq 0$.
Lastly, we show that for all $n \in \mathbb{N}$, $h^{(n)}(0) = 0$. This is done again by induction: when $n = 1$, we have
$$h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = 0$$
When $n = k \geq 2$ where the induction hypothesis holds, we have
$$h^{(k)}(0) = \lim_{x\to 0} \frac{h^{(k-1)}(x) - h^{k-1}(0)}{x - 0} = \lim_{x\to 0} \frac{1}{x} P_{k-1}(\frac{1}{x})h(x) = 0$$
where P_{k-1} is some polynomial. The result follows by induction.

b. Suppose $x \neq 0$. By (i), the existence of $h^{(n)}(0)$ for all $n \in \mathbb{N}$ shows that h is a smooth function on $\mathbb{R}$. In particular, fixing $n \in \mathbb{N}$, we have that $h, \cdots, h^{(n-1)}$ are continuous on $[0, x]$ and $h^{(n)}$ is differentiable on $(0, x)$. Hence, by the Taylor's Theorem, there exists $\xi_n \in (0, x)$ such that

$$h(x) - h(0) = \sum_{i=1}^{n-1} \frac{h^{(i)}(0)}{i!} x^i + \frac{h^{(n)}(\xi_n)}{n!} x^n$$

By definition, the term $\frac{h^{(n)}(\xi_n)}{n!} x^n$ is the remainder term. By (i), we have that $h(x) = \frac{h^{(n)}(\xi_n)}{n!} x^n$ for all $n \in \mathbb{N}$. Hence the sequence of remainder terms $(\frac{h^{(n)}(\xi_n)}{n!} x^n)$ is a constant sequence (with value $h(x)$). In particular $\lim_n \frac{h^{(n)}(\xi_n)}{n!} x^n = h(x) = e^{-1/x^2} \neq 0$. Hence, the remainder term does not converge to 0.

Comment.

1. The function h in this question is the standard example of a function that is infinitely differentiable at a point but is not analytic at it (a function $f : \mathbb{R} \to \mathbb{R}$ is analytic at a point $p \in \mathbb{R}$ if there exists some neighborhood of p on which the value of f and its Taylor's series at p coincides). You will learn more on that in the later part of the course.

2. Alternatively, one can show that $\lim_{x \to 0} h(x)/x^k$ for all $k \in \mathbb{N}$ by considering Taylor's Theorem on the exponential function:
   Since the exponential function is smooth, for all $k \in \mathbb{N}$ and $x > 0$, by Taylor's Theorem, there exists $\xi \in (0, x)$ such that the following inequality can be established.

   $$e^x = \sum_{n=0}^{k} \frac{x^n}{n!} + e^{\xi} \frac{x^{k+1}}{(k+1)!} > \sum_{n=0}^{k} \frac{x^n}{n!} > \frac{x^k}{k!}$$

   Hence for all $x \neq 0$, we have $x^{-2} > 0$. Therefore, $e^{x^{-2}} > \frac{x^{-2k}}{k!}$ for all $k \in \mathbb{N}$, in particular, we have

   $$\frac{h(x)}{x^k} = \frac{e^{-x^{-2}}}{x^k} < \frac{1}{x^k} \frac{k!}{x^{-2k}} = k! x^k$$

   for all $k \geq 1$. It is then clear that $\lim_{x \to 0} \frac{h(x)}{x^k} = 0$ by Squeeze Theorem.

3. To proceed after showing that $\lim_{x \to 0} \frac{h(x)}{x^k} = 0$, one can instead apply induction to show that $\lim_{x \to 0} \frac{h^{(n)}(x)}{x^k} = 0$ for all $n, k \in \mathbb{N}$. In the progress, the nth order Leibniz's Rule would be used on $h^{(n+1)}$ by writing it as the nth derivative of $h^{(1)}$. This way, one does not have to identify the form of $h^{(n)}$ like the solution above, and once such fact has been done the final result that $h^{(n)}(0) = 0$ is clear (by expanding whose definition with an induction proof). Interested readers may try to find this alternative way doing the quesiton by following the above hints.

4. One cannot claim $h^{(k)}(0) = 0$ by using $\lim_{x \to 0} h^{(k)}(x) = 0$ without first showing the continuity of the latter at 0.