MATH 2060B - HW 1 - Solutions[1]

1 (P.171 Q4)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$.

a. Show that f is differentiable at $x = 0$

b. Find $f'(0)$

Solution.

a. By definition of differentiability, it suffices to verify the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$.
Let $\epsilon > 0$. Take $\delta := \epsilon > 0$. Now suppose $0 < |x - 0| < \delta$. By definition of f, we have

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \frac{|f(x)|}{|x|} \leq \frac{\max\{|x^2|, 0\}}{|x|} = |x| < \delta = \epsilon$$

Hence by the $\epsilon - \delta$ definition, the limit is verified.

b. By definition, the derivative at $x = 0$ is given by

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

Hence, $f'(0) = 0$ by the first part.

Comment.

a. It is not accepted to compute $\lim_{\substack{x\to 0 \\ x\in\mathbb{Q}}} \frac{f(x) - f(0)}{x - 0}$ and $\lim_{\substack{x\to 0 \\ x\notin\mathbb{Q}}} \frac{f(x) - f(0)}{x - 0}$ and claim the existence of the limit in question without verifying why the equality between them gives the answer.

b. -

[1] Please feel free to email your TA at kllam@math.cuhk.edu.hk for any questions concerning homework.

2 (P.171 Q10)**.** Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \begin{cases} x^2 \sin\left(1/x^2\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$.

a. Show that g is differentiable for all $x \in \mathbb{R}$.

b. Show that the derivative g' is not bounded on the interval $[-1, 1]$

Solution.

a. **Case 1:** Suppose $x \neq 0$. Let $I \subset \mathbb{R}$ be an open interval such that $x \in I$ but $0 \notin I$. Define $f_1, f_2, f_3 : I \to \mathbb{R}$ by $f_1(t) = t^2, f_2(t) = 1/t^2, f_3(t) = \sin\left(1/t^2\right)$. Note that $g = f_1 \cdot f_3$ on I. It suffices to show that $f_1 \cdot f_3$ is differentiable at x. By product rule, it remains to show f_1 and f_3 are differentiable at x individually. Since f_1 is a polynomial from an open set, the result is clear. For f_3, note that $f_3(t) = \sin(f_2(t))$ for $t \in I$. Since $x \neq 0$, $f_1(x) \neq 0$. By quotient rule, since f_1 is differentiable at x,$f_2(t) = 1/f_1(t)$ is differentiable at x. Furthermore since $t \mapsto \sin(t)$ is differentiable everywhere on $\mathbb{R}$, it is differentiable at $f_2(x) = 1/x^2$. By chain rule, $f_3(t) = \sin(f_2(t))$ is differentiable at x.
Case 2: Suppose $x = 0$. Then for all $t \neq 0$, we have

$$\left|\frac{g(t) - g(0)}{t - 0}\right| = \left|t \sin\left(1/t^2\right)\right| \leq |t|$$

By the sandwich theorem, since $\lim_{t\to 0} |t| = 0$, we have $\lim_{t\to 0} \left|\frac{g(t)-g(0)}{t-0}\right| = 0$, which implies $\lim_{t\to 0} \frac{g(t)-g(0)}{t-0} = 0$. By definition of differentiability, g is differentiable at $x = 0$.

b. By chain rule and product rule, we can compute that $g'(x) = \begin{cases} 2x \sin\left(1/x^2\right) - 2x^{-1} \cos\left(1/x^2\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$
Now consider the sequence defined by $x_n := 1/\sqrt{2n\pi}$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, we have $x_n \in [-1, 1]$, $\sin\left(1/x_n^2\right) = 0$ and $\cos\left(1/x_n^2\right) = 1$. Hence, $g'(x_n) = -2\sqrt{2n\pi}$ for all $n \in \mathbb{N}$ and $\lim_n g'(x_n) = -\infty$. Therefore, $(g'(x_n))$ is an unbounded sequence. It is easy to see that the existence of such sequence contradicts the boundedness of g' on the interval in question.

Comment.

a. Differentiability is a local behavior. To check against differentiability at a point, it usually suffices to restrict the function domain to an open interval (or open neighborhood) containing the point. This principle is used in the solution to the case $x \neq 0$.

b. The boundedness of $g'(x)$ on $[-1, 1]$ is equivalent to the boundedness of $2x^{-1} \cos\left(1/x^2\right)$ there. It is incorrect to verify the boundedness of the latter by stating x^{-1} is unbounded while $\cos\left(1/x^2\right)$ is bounded and hence their *product* is unbounded. Consider simply x^{-1} and x. The former is unbounded on $[-1, 1]$ while the latter is bounded on $[-1, 1]$, but their product, which is a constant function, is still bounded.

3 (P.171 Q13)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and $c \in \mathbb{R}$.

a. Suppose f is differentiable at c. Show that $f'(c) = \lim_{n\to\infty}(n(f(c+1/n) - f(c)))$

b. Show with an example of f that the existence of sequential limit in part(a) does not imply the existence of $f'(c)$.

Solution.

a. Note that we have $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$. By sequential criteria of limit, as $\lim_n 1/n = 0$, we have $f'(c) = \lim_n \frac{f(c+1/n)-f(c)}{1/n} = \lim_n(n(f(c+1/n) - f(c)))$.

b. Here we give 2 examples.

   Example 1: Take $f(x) = |x|$ defined on $\mathbb{R}$ and $c = 0$. It is standard that f is not differentiable at c (for example by considering both right-hand and left-hand limits). However, we still have $\lim_n n(f(c+1/n) - f(c)) = \lim_n n(|1/n| - |0|) = \lim_n n/n = 1$.

   Example 2: Let $A := \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. Take f to be the characteristic function of A, χ_A, that is, $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ and take $c = 0$. It is clear that f is not continuous at 0 and hence not differentiable at 0. However, we have $\lim_n n(f(0+1/n) - f(0)) = \lim_n n(f(1/n) - f(0)) = \lim_n n(1-1) = 0$.

Comment.

a. **Sequential criteria** is the keyword.

b. The above Example 1 demonstrate the importance of computing limits in all (2) directions. Besides the absolute value function, functions like the floor and ceiling are also counterexamples. Example 2 demonstrates instead the importance of having enough points to verify convergence: it is too weak to imply the existence of a limit by approaching with just 1 sequence. Functions like the one in Question 1 give good counterexamples as well.