The Chinese University of Hong Kong
Department of Mathematics

MMAT 5140 Probability Theory 2015 - 2016
Suggested Solution to Test

1 mark will be deducted for each wrong important concept even if everything else in a proof/ statement is correct. Methods not written below are also accepted, but they may not have their detailed marks arrangements below.

1. (a) **2 mark** for each correct axiom.
   i. Axiom I: $P(A) \geq 0 \ \forall A \in \mathcal{E}$;
   ii. Axiom II: $P(\Omega) = 1$;
   iii. Axiom III: For **any disjoint sequence** $\{A_j\} \subseteq \mathcal{E}$,
   $$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P\left(A_j\right).$$

   (b) Marks will be deducted depending on how unclear the explanation of the construction of the counterexample.

   The statement if false. An example to this statement:

   Define $\Omega = \{0, 1, 2\}, \mathcal{E} = \mathcal{P}(\Omega)$ and
   $$P(\{0\}) = P(\{1\}) = 0, \quad P(\{2\}) = 1.$$
   Define $A = \{0\}$, $B = \{1\}$, then $P(A \setminus B) = P(B \setminus A) = 0$ but $A \neq B$.

2. **1 mark** will be deducted without explaining the reordering of the indexes **clearly** (if one did intend to reorder it); **2 marks** will be deducted for the lack of consideration of the set $\{X = 0\}$; **2 marks** will be deducted for assuming $c_j \geq 0 \ \forall 1 \leq j \leq N$; mark(s) will be deducted if different cases are considered but some cases are missing, depending on the number of cases missed.

   Set $c_{N+1} = 0$ and $A_{N+1} = \{X = 0\} = \left(\bigcup_{j=1}^{N} A_j\right)^c$. Note that $A_1, \cdots, A_{N+1}$ form a partition of the sample space Ω and
   $$\{X \leq t\} = \bigcup \{A_{\varphi(j)} : c_j \leq t\}.$$
   Hence, the required distribution function is (finite additivity of disjoint sets is used in the last step)
   $$F_X(t) = P(X \leq t) = P\left(\bigcup \{A_{\varphi(j)} : c_j \leq t\}\right) = \sum_{c_j \leq t} P(A_j).$$

3. (a)

$$\begin{aligned} P\left(A_1 \bigcup A_2\right) &= P\left(A_1 \bigcup (A_2 \setminus A_1)\right) \\ &= P(A_1) + P(A_2 \setminus A_1) \\ &\leq P(A_1) + P(A_2). \end{aligned}$$

(b) **4 marks** will be given if the finite case is proven and **2 marks** will be given with the intention of taking limits from the finite case and using the continuity of probabilities respectively; **4 marks** will be given if one attempted to using the probability axioms directly by setting a suitable sequence of sets and **4 marks** will be given to the correct use of axioms.

Set $F_j = E_j \setminus \bigcup_{i=1}^{j-1} E_i$, then $\{F_j\}$ is a disjoint sequence, $F_j \subseteq E_j$ and $\bigcup_{j=1}^{\infty} F_j = \bigcup_{j=1}^{\infty} E_j$. Using the countably additivity of measure,

$$P\left(\bigcup_{j=1}^{\infty} E_j\right) = P\left(\bigcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} P(F_j) \leq \sum_{j=1}^{\infty} P(E_j).$$

4. (a) **4 marks** will be deducted if only the finite case is considered.

A sequence of events $\{A_j\}$ is said to be **independent** if **for any finite subsequence** $\{A_{j_k}\}$, we have

$$P\left(\bigcap A_{j_k}\right) = \prod P(A_{j_k}).$$

(b) **2 marks** will be given if one considered $\bigcup_{j=1}^{\infty} E_j^c = \left(\bigcap_{j=1}^{\infty} E_j\right)^c$. **2 marks** will be given to the **correct** use the independence of events. **2 marks** will be given to the attempt of taking limit **correctly**. **2 marks** will be given to showing that $\prod_{j=1}^{\infty} p_j = 0$ **with clear explanation**.

Note that

$$\bigcup_{j=1}^{\infty} E_j^c = \left(\bigcap_{j=1}^{\infty} E_j\right)^c.$$

Hence,

$$P\left(\bigcup_{j=1}^{\infty} E_j^c\right) = 1 - P\left(\bigcap_{j=1}^{\infty} E_j\right)$$

and it suffices to find the later term. By the continuity of probability,

$$P\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} P\left(\bigcap_{j=1}^{n} E_j\right).$$

Using the independence of the events,

$$P\left(\bigcap_{j=1}^{n} E_j\right) = \prod_{j=1}^{n} p_j.$$

Now, since $p_j < \frac{1}{2}\ \forall j \in \mathbb{N}$,

$$0 < \prod_{j=1}^{n} p_j < \left(\frac{1}{2}\right)^n.$$

The sandwich theorem shows that $\prod_{j=1}^{\infty} p_j$ exists and equals to 0.

5. (a) Since A_1, A_2, A_3 form a partition of Ω, EA_1, EA_2, EA_3 form a partition of E. Recalling the definition of $P(E|A_j)$, then the finite additivity gives

$$P(E) = \sum_{j=1}^{3} P(EA_j) = \sum_{j=1}^{3} P(E|A_j)P(A_j).$$

(b) Using the notation as in the hint, **1 mark** will be given for finding the correct values of $P(A_1), P(A_2), P(E|A_1), P(E|A_2)$ respectively; **2 marks** will be given for finding the correct values of $P(E|A_3)$; **1 mark** will be given if one attempted to use the formula in (a) **meaningfully**; **1 mark** will be given to the correct value of the answer; **5 marks** will be given for calculating the answer using geometric sum directly correctly.

From counting, $P(A_1) = \frac{5}{36}$, $P(A_2) = \frac{3}{36}$, $P(E|A_1) = 1$, $P(E|A_2) = 0$, $P(E|A_3) = P(E)$. Putting all the values in the formula, $P(E) = \frac{5}{8}$.