MATH4010: Functional Analysis: Home Test **Noon, 5 Nov 2020**

Important Notice:

♣ The answer paper **must be submitted before 06 Nov 2020 at 12 noon.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ The answer paper must include your name and student ID.

Answer ALL Questions

1. **(20 points)**

   You can refer to Example 3.18 in the Lecture Note (2 Nov version) for answering this question.
   Let (c_k) be a sequence of real numbers.

   (i) Show that for each $m \in \mathbb{N}$, there is a bounded variation g_m on $[0,1]$ so that

   $$\int_0^1 x^k dg_m(x) = c_k \tag{1}$$

   for all $k = 0, 1, ..., m$.
   (Hint: By applying the Hahn-Banach Theorem and considering a subspace F of $C[0,1]$ spanned by $\{x^k : k = 0, 1..., m\}$.)

   (ii) Show that there is $C > 0$ such that

   $$|\sum_{k=0}^{m} a_k c_k| \leq C \max\{|\sum_{k=0}^{m} a_k x^k| : x \in [0,1]\}$$

   for any real sequence (a_k) and for all $m = 0, 1, 2, ...$ if and only if there is a bounded variation $\phi : [0,1] \to \mathbb{R}$ such that the Eq 1 above holds for all $k = 0, 1, 2, ...$

Answer:

(i): Fix a positive integer m. Let F be a subspace of $C[0,1]$ spanned by $\{x^k : k = 0,1,...,m\}$. Note that $\{x^k : k = 0,1,...,m\}$ is a linearly independent set. Thus, we can define a linear functional $f : F \to \mathbb{R}$ satisfying $f(x_k) = c_k$ for all $k = 0,1,..,m$. F is of finite dimension, so f is bounded. By using the Hahn-Banach Theorem, there is a bounded linear extension T of f defined on $C[0,1]$. Since $C[0,1]^*$ is the space of all bounded variations on $[0,1]$, there is a bounded variation g_m on $[0,1]$ such that

$$T(\xi) = \int_0^1 \xi(x) dg_m(x)$$

for all $\xi \in C[0,1]$. In particular, we have

$$\int_0^1 x^k dg_m(x) = c_k$$

for all $k = 0,1...,m$ as desired.

(ii): Let X be a subspace of $C[0,1]$ spanned by the set $\{x^k : k = 0,1,2...\}$. As $\{x^k : k = 0,1,2...\}$ is a linearly independent set, we define a linear functional h on X such that $h(x^k) = c_k$ for all $k = 0,1,2....$ We first claim that h is bounded on X. In fact, by the assumption, we have

$$|h(\xi)| \leq C\|\xi\|_\infty$$

for all $\xi \in X$, hence $h \in X^*$. By using the Hahn-Banach Theorem again, there is a bounded linear extension $H \in C[0,1]^*$ of h. Therefore, by using the fact that the dual space $C[0,1]^*$ is the space of all bounded variations on $[0,1]$, there is a bounded variation ϕ on $[0,1]$ such that

$$\int_0^1 x^k d\phi(x) = H(x^k) = h(x^k) = c_k$$

for all $k = 0,1,2....$

2. **(20 points)**

Let $\{Q_i\}_{i\in I}$ be a family of uniform bounded projections on a Banach space X, i.e., there is $C > 0$ such that $\|Q_i\| \leq C$ for all $i \in I$. Let $X_i := Q_i(X)$.
Suppose that the union $\bigcup_{i\in I} X_i$ is dense in X and for any pair $i_1, i_2 \in I$, there is $i_3 \in I$ such that $X_{i_1} \cup X_{i_2} \subseteq X_{i_3}$.

(i) Show that for every $\varepsilon > 0$ and for every finite subset A of X, there is $i \in I$ such that $\sup_{x\in A} \|x - Q_i x\| < \varepsilon$.

(ii) Assume that the n-dimensional finite sequence space $\ell_p^{(n)}$ is a closed subspace of X, where $1 \leq p \leq \infty$ (see Example 1.2 in Lecture Note). Show that for any $\varepsilon > 0$, there exist a subspace F_i of X_i for some $i \in I$ and a linear isomorphism T_i from $\ell_p^{(n)}$ onto F_i such that $\|T_i\|\|T_i^{-1}\| < 1 + \varepsilon$.

Answer:
(i): Let $\varepsilon > 0$. Since the union $\bigcup_{i\in I} X_i$ is dense in X, for each element $a \in A$, we can find $x_{i_a} \in X_{i_a}$ for some $i_a \in I$ such that $\|x_{i_a} - a\| < \varepsilon$. A is finite, by the assumption of X_i's, so there is $i \in I$ such that $x_{i_a} \in X_i$ for all $a \in A$. Then $Q_i(x_{i_a}) = x_{i_a}$ for all $a \in A$. Therefore, we have

$$\|Q_i a - a\| \leq \|Q_i a - Q_i x_{i_a}\| + \|x_{i_a} - a\| \leq (1 + C)\varepsilon$$

for all $a \in A$. The result follows.
(ii): Let $\eta > 0$. Let $(e_k)_{k=1}^n$ be the natural base of ℓ_p^n. Then by Part (i), there is Q_i such that $\|Q_i e_k - e_k\| < \frac{\eta}{n}$ for $k = 1, .., n$. Define a linear map $T : \ell_p^{(n)} \to X_i$ by $T(e_k) := Q_i e_k$ for $k = 1, 2, ..., n$. Note that if $x = \sum_{k=1}^n a_k e_k \in \ell_p^n \subseteq X$, we have $|a_k| \leq \|x\|_p = \|x\|$ for all $k = 1, ..., n$. Thus we have

$$\begin{aligned}\|Tx\| = \|T(\sum_{k=1}^n a_k e_k)\| &\leq \|\sum_{k=1}^n a_k(Te_k - e_k)\| + \|\sum_{k=1}^n a_k e_k\| \\ &\leq \sum_{k=1}^n |a_k|\|Q_i e_k - e_k\| + \|x\| \\ &\leq \|x\|\eta + \|x\| = (1 + \eta)\|x\|.\end{aligned}$$

Hence, $\|T\| \leq 1 + \eta$.

On the other hand, for $x = \sum_{k=1}^{n} a_k e_k \in \ell_p^n$, we have

$$\begin{aligned}
\|x\| &= \|\sum_{k=1}^{n} a_k e_k\| \\
&\leq \|\sum_{k=1}^{n} a_k(e_k - Q_i e_k)\| + \|\sum_{k=1}^{n} a_k Q_i e_k\| \\
&\leq \sum_{k=1}^{n} |a_k| \frac{\eta}{n} + \|T(\sum_{k=1}^{n} a_k e_k)\| \\
&\leq \eta\|x\| + \|Tx\|.
\end{aligned}$$

Therefore, we have

$$(1-\eta)\|x\| \leq \|Tx\|$$

for all $x \in \ell_p^{(n)}$. Thus, if $1-\eta > 0$, then $\|T^{-1}\| \leq \frac{1}{1-\eta}$.
Therefore, T is an isomorphism from ℓ_p^n onto a subspace F_i of X_i for some $i \in I$. Moreover, we have $\|T\|\|T^{-1}\| \leq \frac{1+\eta}{1-\eta}$. Note that $\frac{1+\eta}{1-\eta} \to 1+$ as $\eta \to 0+$. Therefore, for any $\varepsilon > 0$, we can choose $0 < \eta <<$, so that $1 < \frac{1+\eta}{1-\eta} < 1+\varepsilon$. Then the proof is complete.

$***$ **END OF PAPER** $***$