Solution 2

Exercise 2.13

Let A be a non-empty subset of X. A point $a \in X$ is called a boundary point of A if $B(a,r) \cap A \neq \emptyset$ and $B(a,r) \cap A^c \neq \emptyset$ for all $r > 0$, where A^c denotes the complement of A in X. The set of all boundary points, write ∂A, of A is called the boundary of A.

(i) Find the boundaries of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}$.

(ii) Let $X = (0,1) \cup (2,3)$. Find the boundary of the set $(0,1)$ in X.

(iii) Show that the boundary ∂A is a closed subset of X.

(iv) Show that $\overline{A} = A \cup \partial A$.

Solution. (i) We show that $\partial \mathbb{Z} = \mathbb{Z}$. Let $x \in \mathbb{Z}$. Then for any $r > 0$,

$$x \in B(x,r) \cap \mathbb{Z} \quad \text{and} \quad x + \min\{r, 1/2\} \in B(x,r) \cap \mathbb{Z}^c.$$

Thus $\mathbb{Z} \subseteq \partial \mathbb{Z}$. On the other hand, if $x \in \mathbb{Z}^c$, then

$$r_0 := \min\{|x-n| : n \in \mathbb{Z}\} > 0,$$

so that $B(x, r_0/2) \cap \mathbb{Z} = \emptyset$, and hence $x \notin \partial \mathbb{Z}$. Therefore $\partial \mathbb{Z} \subseteq \mathbb{Z}$.

Next we show that $\partial \mathbb{Q} = \mathbb{R}$. Let $x \in \mathbb{R}$. By the density of rational and irrational numbers in $\mathbb{R}$, for any $\varepsilon > 0$, there are $p \in \mathbb{Q}$ and $q \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$|x-p| < \varepsilon, \quad |x-q| < \varepsilon.$$

Equivalently, for any $r > 0$,

$$B(x,r) \cap \mathbb{Q} \neq \emptyset, \quad \text{and } B(x,r) \cap \mathbb{Q}^c \neq \emptyset.$$

Thus $x \in \partial \mathbb{Q}$, and hence $\mathbb{R} \subseteq \partial \mathbb{Q}$. $\partial \mathbb{Q} \subseteq \mathbb{R}$ is trivial.

(ii) We show that $\partial(0,1) = \emptyset$ in X. If $x \in (0,1)$, then

$$B(x,1/2) \cap (0,1)^c = B(x,1/2) \cap (2,3) = \emptyset,$$

so that x is not a boundary point of $(0,1)$. If $x \in (2,3)$, then

$$B(x,1/2) \cap (0,1) = \emptyset,$$

so that x is not a boundary point of $(0,1)$. We thus conclude that $\partial(0,1) = \emptyset$ in X.

(iii) From Definition 2.9, ∂A is closed in X if and only if $\overline{\partial A} = \partial A$. From Definition 2.5, $\overline{\partial A} = \partial A \cup D(\partial A)$, where $D(\partial A)$ is the set of all limit points of ∂A, given by

$$D(\partial A) = \{x \in X : (B(x,r) \setminus \{x\}) \cap \partial A \neq \emptyset \text{ for all } r > 0\}. \tag{1}$$

Thus it suffices to show that $D(\partial A) \subseteq \partial A$. Let $x \in D(\partial A)$. Then, since $B(x,r) \supset B(x,r) \setminus \{x\}$, (1) implies that

$$B(x,r) \cap \partial A \neq \emptyset \quad \text{for all } r > 0.$$

Now, for any $r > 0$, there is $x_r \in B(x, r/2) \cap \partial A$, which satisfies

$$B(x_r, s) \cap A \neq \emptyset \quad \text{and} \quad B(x_r, s) \cap A^c \neq \emptyset \quad \text{for all } s > 0.$$

In particular, since $B(x_r, r/2) \subseteq B(x,r)$, we have

$$B(x,r) \cap A \neq \emptyset \quad \text{and } B(x,r) \cap A^c \neq \emptyset$$

for all $r > 0$. Thus $x \in \partial A$. Therefore ∂A is closed.

(iv) Suppose $x \in \partial A$. Then

$$B(x,r) \cap A \neq \emptyset \quad \text{and} \quad B(x,r) \cap A^c \neq \emptyset \quad \text{for all } r > 0. \tag{2}$$

If further $x \notin A$, then (2) implies that

$$(B(x,r) \setminus \{x\}) \cap A \neq \emptyset \quad \text{for all } r > 0.$$

Thus $x \in D(A)$. Therefore $\partial A \setminus A \subseteq D(A)$, and hence $A \cup \partial A \subseteq A \cup D(A) = \overline{A}$.

On the other hand, suppose $x \in D(A)$. Then

$$(B(x,r) \setminus \{x\}) \cap A \neq \emptyset \quad \text{for all } r > 0. \tag{3}$$

If further $x \notin A$, then clearly

$$B(x,r) \cap A^c \neq \emptyset \quad \text{for all } r > 0. \tag{4}$$

Now (3) and (4) together imply that $x \in \partial A$. Therefore $D(A) \setminus A \subseteq \partial A$, whence $\overline{A} = A \cup D(A) \subseteq A \cup \partial A$.

Alternative proof using Proposition 2.6:

By Proposition 2.6, it is obvious that $\partial A \subseteq \overline{A}$. Hence $A \cup \partial A \subseteq A \cup \overline{A} = \overline{A}$.

Suppose $x \in \overline{A}$. By Proposition 2.6 again, we have

$$B(x,r) \cap A \neq \emptyset \quad \text{for all } r > 0.$$

If $x \notin A$, then clearly $B(x,r) \cap A^c \neq \emptyset$ for all $r > 0$. Thus $\overline{A} \setminus A \subseteq \partial A$, whence $\overline{A} \subseteq A \cup \partial A$.

◀