MATH 2050A - HW 5 - Solutions

We would be using the following results.

Lemma 0.1. *(Can be used without proofs.) Let $r \in (0,1)$. Then $\lim_n r^n = 0$.*

Proof. We present here a (smart) proof using subsequence techniques, which have been used by a number of you.
It is easy to see that the sequence (r^n) is decreasing and bounded below. It converges by the monotone convergence theorem. Let $\rho := \lim r^n$. Note that (r^{2n}) is a subsequence of (r^n). Hence $\lim r^{2n} = \lim r^n = \rho$. By commuting limit with (natural number) power, which follows from the product rule of limit, we have

$$\rho^2 = (\lim r^n)^2 = \lim_n (r^n)^2 = \lim_n r^{2n} = \rho$$

Hence $\rho^2 = \rho$. Solving the equation, we have either $\rho = 0$ or $\rho = 1$. It is easy to see the latter is not possible by order limit property and the monotonicity of (r^n). Hence $\lim r^n = \rho = 0$. $\square$

Remark. The binomial theorem is not used here as in previous proofs given by us.

Lemma 0.2. *(Verification is expected.) Let $r \in (0,1)$. Define for all $n \in \mathbb{N}$, $s_n := \sum_{i=0}^n r^i$. Then (s_n) converges and $\sum_{i=0}^\infty r_i := \lim s_n = \lim \sum_{i=0}^n r^i = \frac{1}{1-r}$. Furthermore, $s_n \le \lim s_n$ for all $n \in \mathbb{N}$.*

Proof. Note that for all $n \in \mathbb{N}$, we have $s_n = \sum_{i=0}^n r^i = \frac{r^{n+1}-1}{r-1}$. (This is a simple fact using the (algebraic) distributive law of $\mathbb{R}$ as a field). Hence by algebraic properties of limit, we have $\lim s_n = \frac{\lim r^{n+1}-1}{r-1} = \frac{0-1}{r-1} = \frac{1}{1-r}$.
Observe that (s_n) is increasing. It is also bounded since it converges. The second statement then follows from the bounded monotone convergence theorem that the sequential limit is the supremum of the sequence. $\square$

Solutions

1 (P.91 Q3b)**.** Show directly from the definition that the following is not a Cauchy sequence:

$$\left(n + \frac{(-1)^n}{n}\right)$$

Solution. Let $x_n := (n + (-1)^n/n)$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Without loss of generality, take n to be even. (In fact we can assume n to be in any pre-fixed strictly increasing subset of $\mathbb{N}$). (Why?) Then $n, n+2 \geq n$. It follows that

$$|x_{n+2} - x_n| = \left|n + 2 + \frac{1}{n+2} - n - \frac{1}{n}\right| = \left|2 + \frac{1}{n+2} - \frac{1}{n}\right| \geq |2| - \left|\frac{1}{n+2} - \frac{1}{n}\right| = |2| - \left|\frac{2}{n(n+2)}\right| \geq 2 - 1 = 1$$

The result follows from the negation of the definition of Cauchy sequences.

2 (P.91 Q5)**.** Let $x_n := \sqrt{n}$ for all $n \in \mathbb{N}$. Show that

(i). $\lim_n |x_{n+1} - x_n| = 0$

(ii). (x_n) is *not* a Cauchy sequence.

Solution.

1. Note that for all $n \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| = \left|\sqrt{n+1} - \sqrt{n}\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{2\sqrt{n}}$$

Since $\lim_n 1/2\sqrt{n} = 0$, the result follows by Squeeze Theorem.

2. Let $n \in \mathbb{N}$. Then $2051^2 n, n \geq n$. It follows that

$$|x_{2051^2 n} - x_n| = \left|\sqrt{2051^2 n} - \sqrt{n}\right| = (\sqrt{2051^2} - 1)\sqrt{n} \geq 2050$$

The result follows from the negation of the definition of Cauchy sequences.

3 (P.91 Q9)**.** Let $0 < r < 1$ and (x_n) be a sequence such that $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$. Show that (x_n) is a Cauchy sequence.

Solution. Method 1: Let $\epsilon > 0$. By Lemma 0.1, (r_n) is a convergent sequence and so a Cauchy sequence. (Note that the Cauchy Criteria states the converse; the direction here is easy.) Hence there exists $N \in N$ such that $|r^n - r^m| < \epsilon$ for all $n, m \geq N$. Now suppose $n, m \geq N$. WLOG, take $m > n$, we then have

$$|x_n - x_m| \leq \sum_{i=n}^{m-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{m-1} r^i = r^n \sum_{i=0}^{m-n-1} r^i = r^n \frac{1 - r^{m-n}}{1-r} = \frac{r^n - r^m}{1-r} < \frac{\epsilon}{1-r}$$

The result follows by definition of Cauchy sequences.
Method 2: Let $\epsilon > 0$. Since $r \in (0,1)$, by lemma 0.1, we have $\lim_n r^n = 0$. Take $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|r^n| < \epsilon$. Then for all $n \geq N$ and $p \in \mathbb{N}$, we have

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{n+p-1} r^i = r^n \sum_{i=0}^{p-1} r^i \leq r^n \lim_k \sum_{i=0}^{k} r^i = r^n \frac{1}{1-r} < \frac{\epsilon}{1-r}$$

The first inequality follows from Triangle Inequality while the fourth inequality follows from the convergence of geometric series (see Lemma 0.2). The result follows from the definition of Cauchy sequences.

Remark. Sometimes, the Cauchiness of a sequence (as in Method 1) is already strong enough for us to do questions in the sense that we do not have to know the sequential limit.