MATH 2050A - HW 2 - Solutions

We would be using the following Lemmas.

Lemma 0.1. *Let (x_n) be a sequence of real numbers. Suppose* $\lim x_n$ *exists, then* $\lim x_n = \lim x_{2n} = \lim x_{2n-1}$.

Proof. Let $L := \lim x_n$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geq N$. Note that if $n \geq N$, we have $2n, 2n-1 \geq n \geq N$. Hence, $|x_{2n} - L| < \epsilon$ and $|x_{2n-1} - L| < \epsilon$. Therefore by the $\epsilon - N$ definition, we have $\lim x_{2n} = \lim x_{2n-1} = L = \lim x_n$. □

Solutions

1 (P.61 Q5)**.** Use the definition of the limit of a real sequence to establish the following limits.

a) $\lim \left(\frac{n}{n^2+1}\right) = 0$

b) $\lim \left(\frac{2n}{n+1}\right) = 2$

c) $\lim \left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$

d) $\lim \left(\frac{n^2-1}{2n^2+3}\right) = \frac{1}{2}$

Solution.

(a). Let $\epsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$. Suppose $n \geq N$, then we have

$$\left|\frac{n}{n^2+1} - 0\right| = \left|\frac{n}{n^2+1}\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

The result follows by the $\epsilon - N$ definition.

(b). Let $\epsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ such that $\frac{2}{\epsilon} < N$. Suppose $n \geq N$, then we have

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} \leq \frac{2}{N} < \epsilon$$

The result follows by the $\epsilon - N$ definition.

(c). Let $\epsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ such that $\frac{13}{\epsilon} < N$. Suppose $n \geq N$, then we have

$$\left|\frac{3n+1}{2n+5} - \frac{3}{2}\right| = \left|\frac{13}{2(2n+5)}\right| = \frac{13}{2(2n+5)} < \frac{13}{2n} \leq \frac{13}{n} \leq \frac{13}{N} < \epsilon$$

The result follows by the $\epsilon - N$ definition.

(d). Let $\epsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ such that $\frac{5}{\epsilon} < N$. Suppose $n \geq N$, then we have

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{5}{2(2n^2+3)}\right| = \frac{5}{2(2n^2+3)} < \frac{5}{n^2} \leq \frac{5}{n} \leq \frac{5}{N} < \epsilon$$

The result follows by the $\epsilon - N$ definition.

2 (P.61 Q8)**.** Let (x_n) be a sequence of real numbers.

(i) Prove that $\lim(x_n) = 0$ if and only if $\lim(|x_n|) = 0$

(ii) Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n)

Solution.

1. $(\Rightarrow)$, suppose $\lim(x_n) = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|x_n - 0| < \epsilon$ for all $n \geq N$.
   Hence, if $n \geq N$, we have $||x_n| - 0| = |x_n - 0| < \epsilon$. The result follows by the $\epsilon - N$ definition.
   $(\Leftarrow)$, suppose $\lim(|x_n|) = 0$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $||x_n| - 0| < \epsilon$ for all $n \geq N$
   Hence, if $n \geq N$, we have $|x_n - 0| = ||x_n| - 0| < \epsilon$. The result follows by the $\epsilon - N$ definition.

2. Take $x_n := (-1)^n$ for all $n \in \mathbb{N}$. Since $(|x_n|)$ is a constant sequence, it converges. It remains to show (x_n) does not converge. (We would be giving a proof different from that in the Lecture.) Observe that $x_{2n} = 1$ and $x_{2n-1} = -1$ for all $n \in \mathbb{N}$. Hence $\lim x_{2_n} = 1$ and $\lim x_{2_n - 1} = -1$. So $\lim x_{2_n} \neq \lim x_{2_n-1}$. By the contrapositive of Lemma 0.1, (x_n) diverges.