MATH 2050A - HW 1 - Solutions

We would be using the following Lemmas.

Lemma 0.1. *Let* $A \subset \mathbb{R}$ *be a subset. Suppose* $\max A$ *(resp.* $\min A$*) exists. Then* $\sup A = \max A$ *(resp.* $\inf A = \min A$*).*

Proof. By definition of maximal element, $\max A$ is an upper bound of A.
Let $\epsilon > 0$. Then $\max A - \epsilon < \max A$ while $\max A \in A$ by definition of a maximal element.
Using equivalence definition of supremum, we have $\max A = \sup A$.
The respective result for minimum/infernum follows from that of supremum/maximum by considering $-A$ and $\inf A = -\sup -A$ □

Lemma 0.2. *Let* $\phi \neq A, B \subset \mathbb{R}$ *such that* $A \subset B$*. Suppose* $\sup B$ *exists. Then* $\sup A$ *exists and* $\sup A \leq \sup B$.

Proof. Let $a \in A$. Then $a \in B$.
Since $\sup B$ exists and is an upper bounded of B, we have $a \leq \sup B$.
Therefore, $\sup B$ is an upper bound of A and by Axiom of Completeness, $\sup A$ exists.
Being the least upper bound, $\sup A \leq \sup B$. □

Solutions

1 (P.39 Q4)**.** Let $S_4 := \left\{1 - \dfrac{(-1)^n}{n} | n \in \mathbb{N}\right\}$. Find $\inf S_4$ and $\sup S_4$.

Solution. Define $x_n := 1 - \dfrac{(-1)^n}{n}$ for all $n \in \mathbb{N}$, so $S_4 = \{x_n\}_{n\in\mathbb{N}}$.
Note that for all $k \in \mathbb{N}$, $x_{2k} = 1 - \dfrac{1}{2k} \leq 1$ and $x_{2k-1} = 1 + \dfrac{1}{2k-1} \geq 1$.
Therefore, (x_{2k}) is an increasing sequence bounded above by 1 and (x_{2k-1}) is a decreasing sequence bounded below by 1.
Therefore, we have for all $j, k \in N$.

$$x_2 \leq x_{2j} \leq 1 \leq x_{2k-1} \leq x_1$$

Hence, $x_2 = \min\{x_n\}_{n\in\mathbb{N}} = \min S_4$ and $x_1 = \max\{x_n\}_{n\in\mathbb{N}} = \max S_4$.
By Lemma 0.1, $\inf S_4 = \min S_4 = x_2 = \dfrac{1}{2}$ and $\sup S_4 = \max s_4 = x_1 = 2$

2 (P.39 Q10)**.** Let $A, B \subset \mathbb{R}$ be bounded subsets. Show that

(i) $A \cup B$ is a bounded set.

(ii) $\sup(A \cup B) = \sup\{\sup A, \sup B\}$

Solution. We consider only non-empty A, B.

(i) Since A, B are bounded sets, their supremums and infernums exist.
Let $x \in A \cup B$. Then either $x \in A$ or $x \in B$. If $x \in A$ *(resp.* $x \in B$*)*, then

$$\min\{\inf A, \inf B\} \leq \inf A \text{ (resp. } \inf B) \leq x \leq \sup A \text{ (resp. } \sup B) \leq \max\{\sup A, \sup B\}$$

In either case, $A \cup B$ is both bounded above by $\max\{\sup A, \sup B\}$ and below by $\min\{\inf A, \inf B\}$ and is thus bounded.

(ii) We have already shown in (i) that $A \cup B$ is bounded above by $\max\{\sup A, \sup B\}$, which by Lemma 0.1, is $\sup\{\sup A, \sup B\}$. Thus, $\sup(A \cup B) \leq \sup\{\sup A, \sup B\}$. (Note that $\sup A \cup B$ exists by the Axiom of Copmleteness)
For the other inequality, since $A, B \subset A \cup B$ and $\sup A \cup B$ exists, by Lemma 0.2, we have $\sup A, \sup B \leq \sup A \cup B$. Hence, $\sup\{\sup A, \sup B\} \leq \sup(A \cup B)$
The equality follows since $\leq$ is symmertric.

3 (P.39 Q12)**.** Let $S \subset \mathbb{R}$. Suppose $s^* := \sup S \in S$. Show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$ for all $u \notin S$,

Solution. Observe that the proof of Q2 (ii) is still valid if we only assume A, B are bounded above (instead of being bounded). Therefore, for any non-empty bounded above subsets $\phi \neq A, B$, we still have

$$\sup(A \cup B) = \sup\{\sup A, \sup B\}$$

Back to the question. Let $u \notin S$. Since $\sup S \in S$, $\sup S < \infty$, that is supremum exists for S. So, S is bounded above.

Moreover, it is easy to see that $u = \max\{u\}$ since $\max\{u\} \in \{u\}$. So, by Lemma 0.1, $\sup\{u\} = u$ and so $\{u\}$ is bounded above.

By the above observation on Q2, we have

$$\sup(S \cup \{u\}) = \sup\{\sup S, \sup\{u\}\} = \sup\{s^*, u\}$$